

REPLACING HOMOTOPY ACTIONS BY TOPOLOGICAL ACTIONS, III

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ABSTRACT. For any finite group G , we provide a more elaborate version of Cooke's result in [C] on replacing a homotopy action of G on a space \mathbf{X} by a topological action, in the context of Bredon's equivariant homotopy theory. Our initial data is the diagram of fixed point sets \mathbf{X}^H for each conjugacy class of subgroup $H \leq G$, together with a pointed homotopy action of the group $N_G H/H$ on $\mathbf{X}^H/(\bigcup_{H < K} \mathbf{X}^K)$. We describe an obstruction theory for constructing a suitable diagram $\underline{\mathbf{X}} : \mathcal{O}_G^{\text{op}} \rightarrow \mathcal{T}$ from this data, thus obtaining a G -space $\mathbf{X}' \simeq \mathbf{X}^{\{e\}}$ realizing the given homotopy information. The resulting topological action of G on \mathbf{X}' is determined up to strong (Bredon) G -homotopy type.

INTRODUCTION

A G -space \mathbf{X} is described by a homomorphism $\varphi_{\mathbf{X}} : G \rightarrow \mathcal{A}_{\mathbf{X}}$ into the group of self-homeomorphisms of \mathbf{X} . This $\varphi_{\mathbf{X}}$ induces a homomorphism $G \rightarrow \mathcal{E}_{\mathbf{X}}$ into the group of homotopy classes of self-homotopy equivalences of \mathbf{X} , which defines a *homotopy action* of G on \mathbf{X} .

In [C], Cooke described an obstruction theory for replacing a homotopy action of a discrete group G on a space \mathbf{X} by a continuous action. This theory was extended by Larry Smith (in [Sm]) to diagrams of G -spaces, and the rational case was treated by Oprea in [O]. See also [Z, SV1].

However, the resulting G -space is only determined up to a homotopy equivalence which is a G -map, and in this sense every G -space is equivalent to a *free* one. A more informative version of equivariant homotopy theory, due to Bredon, studies G -spaces \mathbf{X} up to G -homotopy equivalence (that is, G -maps having G -homotopy inverses). This is equivalent to the homotopy theory of diagrams of fixed point sets \mathbf{X}^H for various subgroups $H \leq G$ (see [Br]).

The purpose of this paper is to define the corresponding “Bredon” notion of homotopy action, and describe an inductive process for replacing such an action by a continuous one, together with its associated obstruction theory.

0.1. Homotopy actions. The category of topological spaces will be denoted by \mathcal{T} , and its objects will be denoted by boldface letters: $\mathbf{X}, \mathbf{Y} \dots$. The category of pointed topological spaces $\mathbf{X}_* = (\mathbf{X}, x_0)$ is denoted by \mathcal{T}_* .

For any topological space $\mathbf{X} \in \mathcal{T}$, we let $\mathcal{A}_{\mathbf{X}}$ denote the topological group of self-homeomorphisms of \mathbf{X} , so an action of a (discrete) group G on \mathbf{X} is given by a homomorphism $\varphi_{\mathbf{X}} : G \rightarrow \mathcal{A}_{\mathbf{X}}$, which we call the *action map* of \mathbf{X} . We let $\mathcal{G}_{\mathbf{X}}$ denote the topological monoid of self-homotopy equivalences of \mathbf{X} (sometimes written

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haut \mathbf{X}). This monoid is *group-like*, in the sense that $\mathcal{E}_{\mathbf{X}} := \pi_0 \mathcal{G}_{\mathbf{X}}$ is a group. As noted above, a homomorphism $\alpha : G \rightarrow \mathcal{E}_{\mathbf{X}}$ is called a *homotopy action* of G on \mathbf{X} .

The inclusion $i_{\mathbf{X}} : \mathcal{A}_{\mathbf{X}} \hookrightarrow \mathcal{G}_{\mathbf{X}}$ is a continuous map of associative topological monoids, and we call the composite $\zeta_{\mathbf{X}} := i_{\mathbf{X}} \circ \varphi_{\mathbf{X}} : G \rightarrow \mathcal{G}_{\mathbf{X}}$ the *monoid action map* of \mathbf{X} . If $\mathbf{X}_* = (\mathbf{X}, x_0) \in \mathcal{T}_*$ is a pointed space, $\mathcal{A}_{\mathbf{X}_*}^*$ will denote the topological group of pointed self-homeomorphisms of \mathbf{X}_* . The monoid of pointed self-homotopy equivalences of \mathbf{X}_* is denoted by $\mathcal{G}_{\mathbf{X}_*}^*$, with $\mathcal{E}_{\mathbf{X}_*}^* := \pi_0 \mathcal{G}_{\mathbf{X}_*}^*$, and a pointed action of G on \mathbf{X} , given by the *pointed action map* $\varphi_{\mathbf{X}_*}^* : G \rightarrow \mathcal{A}_{\mathbf{X}_*}^*$, has a *pointed monoid action map* $\zeta_{\mathbf{X}_*}^* := i_{\mathbf{X}_*} \circ \varphi_{\mathbf{X}_*}^* : G \rightarrow \mathcal{G}_{\mathbf{X}_*}^*$. A homomorphism $\alpha^* : G \rightarrow \mathcal{E}_{\mathbf{X}_*}^*$ is called a *pointed homotopy action* of G on the pointed space \mathbf{X}_* .

If we let Λ denote the partially ordered set of subgroups of G , then a *Bredon homotopy action* of G consists of:

- a diagram $\tilde{X} : \Lambda^{\text{op}} \rightarrow \mathcal{T}$;
- For each $H \leq G$, a pointed homotopy action of $N_G H / H$ on the homotopy cofiber \tilde{X}_H^H of the obvious map $\text{hocolim}_{K > H} \tilde{X}(K) \rightarrow \tilde{X}(H)$.

0.2. Lifting problems and obstructions. In order to try to realize such a Bredon homotopy action, we extend Cooke's original approach to realizing ordinary homotopy actions (see Section 2) as follows:

First, we show that a pointed homotopy action $\alpha^* : G \rightarrow \mathcal{E}_{\mathbf{X}_*}^*$ of G on can be realized topologically if and only if we have a lift in:

$$(0.3) \quad \begin{array}{ccc} & & \mathbf{B}\mathcal{G}_{\mathbf{X}_*}^* \\ & \nearrow \Phi^* & \downarrow \mathbf{B}\gamma^* \\ \mathbf{B}G & \xrightarrow{\mathbf{B}\alpha^*} & \mathbf{B}\mathcal{E}_{\mathbf{X}_*}^* \end{array}$$

where $\gamma^* : \mathcal{G}_{\mathbf{X}_*}^* \rightarrow \mathcal{E}_{\mathbf{X}_*}^*$ is the quotient map (see Corollary 3.17).

We therefore have a sequence of obstructions in $H^{n+2}(G; \pi_{n+1}(\mathbf{B}\mathcal{G}_{\mathbf{X}_*}^*))$ to realizing α^* as well as difference obstructions in $H^{n+1}(G; \pi_{n+1}(\mathbf{B}\mathcal{G}_{\mathbf{X}_*}^*))$ for distinguishing between different realizations (cf. Proposition 3.19).

In Section 4, we explain how to transfer a given action along a map $f : \mathbf{X} \rightarrow \mathbf{Y}$. These specific results, not actually needed in this paper, are included for completeness; however, their proofs used in the following section.

Finally, we consider two maps $\mathbf{X} \xrightarrow{f} \mathbf{Y} \xrightarrow{g} \mathbf{Z}$ with given G -actions on \mathbf{X} and \mathbf{Z} , for which we want to “interpolate” a compatible G -action on \mathbf{Y} . For this purpose, we construct a certain homotopy limit $\mathcal{Q}_{f,g}$ (cf. §5.4), and show that we have an such an interpolated G -action if and only if there is a homotopy lifting Ψ in:

$$\begin{array}{ccc} & & \mathbf{B}\mathcal{Q}_{f,g} \\ & \nearrow \Psi & \downarrow \rho \\ \mathbf{B}G & \xrightarrow{(\mathbf{B}\zeta_{\mathbf{X}}, \mathbf{B}\zeta_{\mathbf{Z}})} & \mathbf{B}\mathcal{G}_{\mathbf{X}} \times \mathbf{B}\mathcal{G}_{\mathbf{Z}} \end{array}$$

(see Proposition 5.6). Again this yields a sequence of obstructions to such a lifting in $H^{n+2}(G; \pi_{n+1}F)$ (where F is the homotopy fibre of the map $\rho : \mathbf{B}\mathcal{Q}_{f,g} \rightarrow$

$\mathbf{BG}_{\mathbf{X}} \times \mathbf{BG}_{\mathbf{Z}}$), as well as difference obstructions in $H^{n+1}(G; \pi_{n+1}F)$ (see Proposition 5.14).

0.4. The inductive process. Now, given a Bredon homotopy action as above, we can try to realize it by a topological action using descending induction on the subgroups of G .

Thus (without specifying the G -space \mathbf{X} itself), we assume that for some $H \leq G$ we have constructed the diagram of “fixed point sets” \mathbf{X}^K for all subgroups K of G properly containing H , together with the inclusions $i^* : \mathbf{X}^L \hookrightarrow \mathbf{X}^K$ for $i : K \hookrightarrow L$ and the action of G by conjugation.

- (i) First, we use the obstruction theory of Proposition 3.19 to try to lift the given pointed homotopy action of $N_G H/H$ on $\tilde{\mathbf{X}}_H^H$ to a pointed topological action on a space $\mathbf{X}_H^H \simeq \tilde{\mathbf{X}}_H^H$.
- (ii) If we set $\mathbf{X}_H := \bigcup_{H < K} \mathbf{X}^K$, then we have an action of $N_G H/H$ on \mathbf{X}_H . Moreover, we may assume by induction that this fits into a homotopy cofibration sequence $\mathbf{X}_H \rightarrow \tilde{\mathbf{X}}(H) \rightarrow \mathbf{X}_H^H$, so we may use the obstruction theory of Proposition 5.14 to try and interpolate an action of $N_G H/H$ on a new space $\mathbf{X}^H \simeq \tilde{\mathbf{X}}(H)$. If we succeed, we have extended our diagram of “fixed point sets” to H , too.

In Theorem 6.11 below we show that if this process can be completed for all $H \leq G$, we obtain a G -space realizing the given Bredon homotopy action. Thus the obstructions of Propositions 3.19 and 5.14 are the only ones to realizing a Bredon homotopy action. Moreover, the difference obstructions distinguish between the resulting realizations up to G -homotopy equivalence.

0.5. Remark. An alternative approach to realizing “Bredon” homotopy actions is to use the obstruction theory of Dwyer, Kan, and Smith for rectifying general homotopy-commutative diagrams (cf. [DKS2]). Thus we could start with a functor $\mathcal{O}_G^{\text{op}} \rightarrow \text{ho } \mathcal{T}$, and try to lift it to a functor $\mathcal{O}_G^{\text{op}} \rightarrow \mathcal{T}$ (which then determines a G -space up to G -homotopy equivalence – see §1.3).

However, this requires more complicated initial data than the approach taken here, and the successive obstructions to rectification lie in the $(\mathcal{S}, \mathcal{O})$ -cohomology groups of the diagram as a whole (cf. [DKS1, §2]), rather than in ordinary group cohomology, as in our case.

1. G -SPACES AND THE ORBIT CATEGORY

Let G be a fixed group. Bredon’s approach to G -equivariant homotopy theory (cf. [Br, E]) reduces the study of a G -space \mathbf{X} to the system of fixed point sets under the subgroups of G . To describe it, we recall the following:

1.1. G -spaces. A G -space is a topological space \mathbf{X} equipped with a left G -action. The category of G -spaces with G -maps (i.e., G -equivariant continuous maps) will be denoted by $G\text{-}\mathcal{T}$. We write \mathbf{X}^H for the *fixed point set* $\{x \in \mathbf{X} : hx = x \ \forall h \in H\}$ of \mathbf{X} under a subgroup $H \leq G$.

An important example is a G -CW complex, obtained from a disjoint union of basic G -sets by attaching G -cells of the form $G/H \times \mathbf{D}^{n+1}$ (see [I]). For finite G , this is equivalent to \mathbf{X} being a CW-complex on which G acts cellularly (see [tD]).

1.2. The orbit category. A *basic G -set* is the set of left cosets G/H for some subgroup $H \leq G$, with the left G -action. The *orbit category* \mathcal{O}_G of G has the basic G -sets as objects, and G -equivariant maps as morphisms.

Any map $G/H \rightarrow G/K$ in \mathcal{O}_G can be factored as $i_* : G/H \rightarrow G/K^{a^{-1}}$ (induced by the inclusion $i : H \hookrightarrow K^{a^{-1}}$), followed by an isomorphism $\phi_a^{K^{a^{-1}}} : G/K^{a^{-1}} \rightarrow G/K$, where $K^{a^{-1}} = aKa^{-1}$, $a \in G$, and $\phi_a^{K^{a^{-1}}}$ is induced by the right translation ($g \mapsto ag$). This map can also be decomposed as $i_*^a \circ \phi_a^H$, where $i^a : H^a \hookrightarrow K$ is the conjugate of $i : H \hookrightarrow K^{a^{-1}}$ by a . Two maps $\phi_a^{K^{a^{-1}}} \circ i_*$ and $\phi_b^{K^{b^{-1}}} \circ j_*$ from G/H to G/K are the same in \mathcal{O}_G if and only if $a^{-1}b \in K$. Thus the automorphism group $W_H := \text{Aut}_{\mathcal{O}_G}(G/H)$ of $G/H \in \mathcal{O}_G$ is $N_G H/H$ (where $N_G H$ is the normalizer of H in G).

We shall be mainly concerned with the opposite category $\mathcal{O}_G^{\text{op}}$ of the orbit category: it has the same objects, but its morphisms are generated by $(\phi_a^H)^{\text{op}} : G/H^a \rightarrow G/H$ and $i^* : G/K \rightarrow G/H$ for $i : H \hookrightarrow K$, with the same the automorphism group W_H for $G/H \in \mathcal{O}_G^{\text{op}}$.

1.3. \mathcal{O}_G -diagrams. An $\mathcal{O}_G^{\text{op}}$ -*diagram* in \mathcal{T} is a functor $\underline{X} : \mathcal{O}_G^{\text{op}} \rightarrow \mathcal{T}$, and the category of all such diagrams will be denoted by $\mathcal{T}^{\mathcal{O}_G^{\text{op}}}$. Since \mathcal{T} is a simplicial model category (cf. [Q, II, §2]), $\mathcal{T}^{\mathcal{O}_G^{\text{op}}}$ has a projective simplicial model category structure in which a map $f : \underline{X} \rightarrow \underline{Y}$ of $\mathcal{O}_G^{\text{op}}$ -diagrams is a weak equivalence (respectively, a fibration) if for each $H \leq G$, $f(G/H) : \underline{X}(G/H) \rightarrow \underline{Y}(G/H)$ is a weak equivalence (respectively, a fibration). The mapping spaces are defined using the simplicial structure in \mathcal{T} by $\text{Map}_{\mathcal{T}^{\mathcal{O}_G^{\text{op}}}}(\underline{X}, \underline{Y})_n := \text{Hom}_{\mathcal{C}^{\mathcal{O}_G^{\text{op}}}}(\underline{X} \otimes \Delta[n], \underline{Y})_n$ (with a trivial G -action on $\Delta[n]$).

There is an analogous simplicial model category structure on $G\text{-}\mathcal{T}$, in which a G -map $f : \mathbf{X} \rightarrow \mathbf{Y}$ is a weak equivalence (respectively, fibration) if for each $H \leq G$, the map $f|_{\mathbf{X}^H}$ is a weak equivalence (respectively, fibration). See [DK] and compare [Pi].

Note that the fixed point set functor $\Phi : G\text{-}\mathcal{T} \rightarrow \mathcal{T}^{\mathcal{O}_G^{\text{op}}}$, sending a G -space \mathbf{X} to the diagram $\Phi\mathbf{X} : \mathcal{O}_G^{\text{op}} \rightarrow \mathcal{T}$ defined:

$$(1.4) \quad (\Phi\mathbf{X})(G/H) := \mathbf{X}^H$$

has a right adjoint $\Psi : \mathcal{T}^{\mathcal{O}_G^{\text{op}}} \rightarrow G\text{-}\mathcal{T}$ (see [E, Theorem 1]). In fact, this adjoint pair constitutes a simplicial Quillen equivalence between $G\text{-}\mathcal{T}$ and $\mathcal{T}^{\mathcal{O}_G^{\text{op}}}$.

1.5. The Borel construction. For a group G , let $\mathbf{E}G$ denote any contractible space with a free G -action, and $\mathbf{B}G = \mathbf{E}G/G$ the classifying space of G . For any G -space \mathbf{X} , the associated free G -space is $\mathbf{E}G \times \mathbf{X}$ with diagonal G -action. The *Borel construction* on \mathbf{X} is the (homotopy) quotient $X_{hG} := \mathbf{E}G \times_G \mathbf{X}$ – that is, the orbit space of $\mathbf{E}G \times \mathbf{X}$.

For any G -space \mathbf{X} , the projection $p_{\mathbf{X}} : \mathbf{E}G \times \mathbf{X} \rightarrow \mathbf{X}$ is a G -map which is a homotopy equivalence, and for any point $e_0 \in \mathbf{E}G$, the inclusion $i_{\mathbf{X}} = (c_{e_0}, \text{Id}_{\mathbf{X}}) : \mathbf{X} \hookrightarrow \mathbf{E}G \times \mathbf{X}$ is also a homotopy equivalence (though not a G -map), with $p_{\mathbf{X}} \circ i_{\mathbf{X}} = \text{Id}_{\mathbf{X}}$.

1.6. Definition. A G -map $h : \mathbf{X} \rightarrow \mathbf{Y}$ which at the same time is a (non-equivariant) homotopy equivalence will be called a *Borel G -equivalence*. If $x_0 \in \mathbf{X}$ and $y_0 = h(x_0) \in \mathbf{Y}$ are G -base-points (fixed under the G action) and h is a pointed homotopy equivalence, it will be called a *pointed Borel G -equivalence*.

1.7. Lemma. *For any homotopy equivalence $h : \mathbf{X} \rightarrow \mathbf{Y}$ between CW complexes, there is a CW complex \mathbf{Z} with homotopy equivalences $i : \mathbf{X} \rightarrow \mathbf{Z}$ and $i' : \mathbf{Y} \rightarrow \mathbf{Z}$ such that $i \sim h \circ i'$, inducing strictly multiplicative monic homotopy equivalences $i_* : \mathcal{G}_{\mathbf{X}} \rightarrow \mathcal{G}_{\mathbf{Z}}$ and $i'_* : \mathcal{G}_{\mathbf{Y}} \rightarrow \mathcal{G}_{\mathbf{Z}}$.*

Proof. Factoring h as $p' \circ i = h$ with i a cofibration and p' a fibration, and using the cofibrancy of \mathbf{X} and \mathbf{Y} , we obtain a diagram of homotopy equivalences

$$(1.8) \quad \begin{array}{ccc} & \mathbf{Z} & \\ \overset{p}{\curvearrowright} & & \overset{i'}{\curvearrowright} \\ \mathbf{X} & \xrightarrow{i} & \mathbf{Y} \\ & \xrightarrow{h} & \\ & \xleftarrow{p'} & \end{array}$$

with $p \circ i = \text{Id}_{\mathbf{X}}$ and $p' \circ i' = \text{Id}_{\mathbf{Y}}$. Define $i_* : \mathcal{G}_{\mathbf{X}} \rightarrow \mathcal{G}_{\mathbf{Z}}$ by $\varphi \mapsto i \circ \varphi \circ p$ (for any homotopy equivalence $\varphi : \mathbf{X} \rightarrow \mathbf{X}$). Because $p \circ i = \text{Id}_{\mathbf{X}}$, the map i_* is monic, preserves compositions, and has a (non-monoidal) homotopy inverse $p_* : \mathcal{G}_{\mathbf{Z}} \rightarrow \mathcal{G}_{\mathbf{X}}$. Similarly for i'_* . \square

1.9. Remark. Any homotopy equivalence $h : \mathbf{X} \rightarrow \mathbf{Y}$ induces a homotopy equivalence $\mathbf{B}\mathcal{G}_{\mathbf{X}} \simeq \mathbf{B}\mathcal{G}_{\mathbf{Y}}$ (cf. [F1, Satz 7.7]). In fact, we can apply the classifying space functor \mathbf{B} to the maps i_* and i'_* , obtaining homotopy equivalences:

$$\mathbf{B}\mathcal{G}_{\mathbf{X}} \xrightarrow{\mathbf{B}i_*} \mathbf{B}\mathcal{G}_{\mathbf{Z}} \xrightarrow{(\mathbf{B}i'_*)^{-1}} \mathbf{B}\mathcal{G}_{\mathbf{Y}}.$$

The pointed versions of these statements also hold.

2. REALIZING HOMOTOPY ACTIONS

The basic notion of a homotopy action of a (discrete) group G on a space \mathbf{X} is given by a homomorphism $\alpha : G \rightarrow \mathcal{E}_{\mathbf{X}}$ into the group of homotopy classes of self-homotopy equivalences of \mathbf{X} (see §0.1).

In this section we assume all (pointed) spaces are (pointed) CW-complexes, and all G -spaces are G -CW complexes.

2.1. Definition. A *rectification* (or *realization*) of a homotopy action $\alpha : G \rightarrow \mathcal{E}_{\mathbf{X}}$ as above consists of a homotopy equivalence $h : \mathbf{Y} \rightarrow \mathbf{X}$ and a homomorphism $\varphi : G \rightarrow \mathcal{A}_{\mathbf{Y}}$ making the diagram of H -space maps of associative (topological) monoids:

$$(2.2) \quad \begin{array}{ccccc} & & \mathcal{A}_{\mathbf{Y}} & \xrightarrow{\quad} & \mathcal{G}_{\mathbf{Y}} & \xrightarrow{h_*} & \mathcal{G}_{\mathbf{X}} \\ & \nearrow \varphi & & & & & \downarrow \gamma \\ G & \xrightarrow{\quad \alpha \quad} & & & \mathcal{E}_{\mathbf{X}} & . \end{array}$$

commute (in the notation of §0.1). Here $\gamma : \mathcal{G}_{\mathbf{X}} \rightarrow \mathcal{E}_{\mathbf{X}}$ is the map of associative topological monoids given by the quotient map $\mathcal{G}_{\mathbf{X}} \rightarrow \pi_0 \mathcal{G}_{\mathbf{X}} = \mathcal{E}_{\mathbf{X}}$.

2.3. Remark. Applying the classifying space functor \mathbf{B} to (2.2), using Remark 1.9, and inverting the homotopy equivalence $\mathbf{B}h_*$ yields a diagram:

$$(2.4) \quad \begin{array}{ccc} & \mathbf{B}\mathcal{G}_{\mathbf{X}} & \\ \nearrow \Phi & & \downarrow \mathbf{B}\gamma \\ \mathbf{B}G & \xrightarrow{\mathbf{B}\alpha} & \mathbf{B}\mathcal{E}_{\mathbf{X}} , \end{array}$$

which commutes up to homotopy if and only if $\pi_1\Phi = \alpha$ as a homomorphism: $G = \pi_1\mathbf{B}G \rightarrow \mathcal{E}_{\mathbf{X}} = \pi_1\mathbf{B}\mathcal{E}_{\mathbf{X}} = \pi_1\mathbf{B}\mathcal{G}_{\mathbf{X}}$ – that is, if and only if (2.2) commutes. Note that Φ is just $\mathbf{B}\zeta_{\mathbf{Y}}$ (\mathbf{B} of the monoid action map), up to homotopy.

Cooke’s method for lifting a homotopy action to a topological action is based on the following:

2.5. Proposition ([C, Theorem 1.1(a)]). *A homotopy action $\alpha : G \rightarrow \mathcal{E}_{\mathbf{X}}$ can be rectified if and only if there is a map $\Phi : \mathbf{B}G \rightarrow \mathbf{B}\mathcal{G}_{\mathbf{X}}$ making (2.4) commute up to homotopy.*

We give a proof of this result in order to provide the necessary background for the variants we shall need later:

Proof. If the homotopy action can be rectified, by Remark 2.3 we obtain a lift (2.4).

Conversely, given a lift Φ in (2.4), the inclusion $j : \mathcal{G}_{\mathbf{X}_*}^* \hookrightarrow \mathcal{G}_{\mathbf{X}}$ fits into a homotopy fibration sequence:

$$(2.6) \quad \Omega\mathbf{X} \rightarrow \mathcal{G}_{\mathbf{X}_*}^* \xrightarrow{j} \mathcal{G}_{\mathbf{X}} \xrightarrow{\text{ev}_{x_0}} \mathbf{X} \xrightarrow{k} \mathbf{B}\mathcal{G}_{\mathbf{X}_*}^* \xrightarrow{\mathbf{B}j} \mathbf{B}\mathcal{G}_{\mathbf{X}},$$

where $\mathbf{B}j$ is universal for Hurewicz fibrations with homotopy fiber \mathbf{X} (cf. [A, St], and see [BGM, Theorem 5.6] & [DZ, Proposition 4.1]). Therefore, pulling back the fibration $\mathbf{B}j$ along Φ yields a fibration sequence

$$\mathbf{X}_1 \hookrightarrow E_\theta \xrightarrow{\theta} \mathbf{B}G,$$

with $\mathbf{X}_1 \simeq \mathbf{X}$. On the other hand, pulling back the universal principal G -bundle $G \hookrightarrow \mathbf{E}G \twoheadrightarrow \mathbf{B}G$ along θ , we obtain a commuting diagram with rows and columns all fibration sequences:

$$(2.7) \quad \begin{array}{ccccc} * & \xrightarrow{\quad} & G & \xrightarrow{\quad=} & G \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{X}_1 & \xrightarrow{\quad\cong\quad} & \mathbf{X}' & \xrightarrow{\quad\text{PB}\quad} & \mathbf{E}G \\ \downarrow & & \downarrow \xi & & \downarrow \\ \mathbf{X}_1 & \xrightarrow{\quad=} & E_\theta & \xrightarrow{\quad\theta\quad} & \mathbf{B}G. \end{array}$$

Since the middle column is a principle G -bundle, \mathbf{X}' has a free G -action, and since $\mathbf{E}G$ is contractible, the fiber \mathbf{X}_1 is homotopy equivalent to \mathbf{X}' , but also to \mathbf{X} by the universal property of $\mathbf{B}j$.

Applying Kan’s G -functor to $\mathbf{B}G \xrightarrow{\Phi} \mathbf{B}\mathcal{G}_{\mathbf{X}} \xleftarrow{\mathbf{B}j} \mathbf{B}\mathcal{G}_{\mathbf{X}_*}^*$ and realizing yields two homomorphisms of topological groups: $\widehat{G} \xrightarrow{\widehat{\zeta}} \widehat{\mathcal{G}}_{\mathbf{X}} \xleftarrow{\widehat{j}} \widehat{\mathcal{G}}_{\mathbf{X}_*}^*$, with homotopy equivalences $G \simeq \Omega\mathbf{B}G \simeq \widehat{G}$, $\mathcal{G}_{\mathbf{X}} \simeq \Omega\mathbf{B}\mathcal{G}_{\mathbf{X}} \simeq \widehat{\mathcal{G}}_{\mathbf{X}}$, and $\mathcal{G}_{\mathbf{X}_*}^* \simeq \Omega\mathbf{B}\mathcal{G}_{\mathbf{X}_*}^* \simeq \widehat{\mathcal{G}}_{\mathbf{X}_*}^*$ (since $\mathcal{G}_{\mathbf{X}}$ and $\mathcal{G}_{\mathbf{X}_*}^*$ are grouplike topological monoids).

Replacing $\mathbf{B}\widehat{j}$ by a fibration $\widehat{\mathbf{B}}\widehat{j} : \widehat{\mathbf{B}}\widehat{\mathcal{G}}_{\mathbf{X}_*}^* \rightarrow \mathbf{B}\widehat{\mathcal{G}}_{\mathbf{X}_*}^*$ with fiber $\widehat{\mathbf{X}} \simeq \mathbf{X}$, we pull back the universal principle $\widehat{\mathcal{G}}_{\mathbf{X}}$ -bundle:

$$(2.8) \quad \widehat{\mathcal{G}}_{\mathbf{X}} \hookrightarrow \mathbf{E}\widehat{\mathcal{G}}_{\mathbf{X}} \twoheadrightarrow \mathbf{B}\widehat{\mathcal{G}}_{\mathbf{X}}.$$

along $\widehat{\mathbf{B}}\widehat{j}$ to obtain a principle $\widehat{\mathcal{G}}_{\mathbf{X}}$ -bundle

$$(2.9) \quad \widehat{\mathcal{G}}_{\mathbf{X}} \hookrightarrow \widehat{\mathbf{X}}'' \xrightarrow{\eta} \widehat{\mathbf{B}\mathcal{G}_{\mathbf{X}_*}^*}$$

which is just the (looped) universal Hurewicz fibration (2.6), up to homotopy, so $\widehat{\mathbf{X}}'' \simeq \mathbf{X}$.

Similarly, pulling back $\widetilde{\mathbf{B}}\widehat{j} : \widetilde{\mathbf{B}}\widehat{\mathcal{G}}_{\mathbf{X}^*}^* \rightarrow \mathbf{B}\widehat{\mathcal{G}}_{\mathbf{X}}$, along $\mathbf{B}\widehat{\zeta} : \mathbf{B}\widehat{G} \rightarrow \mathbf{B}\widehat{\mathcal{G}}_{\mathbf{X}}$ (which is just Φ , up to homotopy), we obtain a fibration $\widehat{\theta} : E_{\widehat{\theta}} \rightarrow \mathbf{B}\widehat{G}$, which is just the above map $\theta : E_{\theta} \rightarrow \mathbf{B}G$, up to homotopy.

Finally, pulling back the universal principle \widehat{G} -bundle $\widehat{G} \rightarrow \mathbf{E}\widehat{G} \rightarrow \mathbf{B}\widehat{G}$ along $\widehat{\theta} : E_{\widehat{\theta}} \rightarrow \mathbf{B}\widehat{G}$ we obtain a principle \widehat{G} -bundle $\widehat{G} \hookrightarrow \widehat{\mathbf{X}}' \rightarrow E_{\widehat{\theta}}$, which is $G \hookrightarrow \mathbf{X}' \rightarrow E_{\theta}$, up to homotopy.

All these fit into a commutative diagram of vertical principle bundles:

$$(2.10) \quad \begin{array}{ccccccc} \widehat{G} & \xrightarrow{=} & \widehat{G} & & \widehat{G} & & \widehat{G} \\ \downarrow & \searrow^{\widehat{\zeta}} & \downarrow & & \downarrow & \searrow^{\widehat{\zeta}} & \downarrow \\ \widehat{\mathbf{X}}' & \xrightarrow{\quad} & \mathbf{E}\widehat{G} & & \widehat{\mathcal{G}}_{\mathbf{X}} & \xrightarrow{=} & \widehat{\mathcal{G}}_{\mathbf{X}} \\ \downarrow \xi & \searrow^{\widehat{\zeta}} & \downarrow & \searrow^{h} & \downarrow \mathbf{E}\widehat{\zeta} & \searrow^{\quad} & \downarrow \\ E_{\widehat{\theta}} & \xrightarrow{\widehat{\theta}} & \mathbf{B}\widehat{G} & & \widehat{\mathbf{X}}'' & \xrightarrow{\quad} & \mathbf{E}\widehat{\mathcal{G}}_{\mathbf{X}} \\ & \searrow^g & & \searrow^{\mathbf{B}\widehat{\zeta}} & \downarrow \eta & \searrow^{\quad} & \downarrow \\ & & \widetilde{\mathbf{B}}\widehat{\mathcal{G}}_{\mathbf{X}^*}^* & \xrightarrow{\widetilde{\mathbf{B}}\widehat{j}} & \mathbf{B}\widehat{\mathcal{G}}_{\mathbf{X}} & & \mathbf{B}\widehat{\mathcal{G}}_{\mathbf{X}} \end{array}$$

in which the map h is a homotopy equivalence, induced by the structure map g of the bottom pull-back square (since $\mathbf{B}\widehat{\zeta}$ classifies the fibration $\widehat{\theta}$ up to homotopy).

Thus we see that the \widehat{G} -action on $\widehat{\mathbf{X}}'$ (in the principle bundle), corresponds under the homotopy equivalence $h : \widehat{\mathbf{X}}' \rightarrow \widehat{\mathbf{X}}''$ to the $\widehat{\mathcal{G}}_{\mathbf{X}}$ -action on $\widehat{\mathbf{X}}''$ via the homomorphism of topological groups $\widehat{\zeta} : \widehat{G} \rightarrow \widehat{\mathcal{G}}_{\mathbf{X}}$. This means that if $\zeta_{\widehat{\mathbf{X}}'} : \widehat{G} \rightarrow \mathcal{A}_{\widehat{\mathbf{X}}'} \hookrightarrow \mathcal{G}_{\widehat{\mathbf{X}}'}$ describes the first action, and the H -map homotopy equivalence $\zeta_{\widehat{\mathbf{X}}''} : \widehat{\mathcal{G}}_{\mathbf{X}} \rightarrow \mathcal{A}_{\widehat{\mathbf{X}}''} \hookrightarrow \mathcal{G}_{\widehat{\mathbf{X}}''}$ describes the second, then

$$(2.11) \quad \zeta_{\widehat{\mathbf{X}}''} \circ \widehat{\zeta} \sim h_* \circ \zeta_{\widehat{\mathbf{X}}'}.$$

Applying the homotopy equivalences $\widehat{\mathbf{X}}' \simeq \mathbf{X} \simeq \mathbf{X}'$, $\widehat{G} \simeq G$, and $K \simeq \mathcal{G}_{\mathbf{X}} \simeq \mathcal{G}_{\mathbf{X}'} \simeq \mathcal{G}_{\widehat{\mathbf{X}}'}$, and recalling that Φ is represented up to homotopy by $\mathbf{B}\widehat{\zeta}$ – that is,

$$G = \pi_1 \mathbf{B}\widehat{G} \xrightarrow{(\mathbf{B}\widehat{\zeta})_*} \pi_1 \mathbf{B}\widehat{\mathcal{G}}_{\mathbf{X}} = \pi_1 \mathcal{G}_{\mathbf{X}} = \mathcal{E}_{\mathbf{X}}$$

is the given map α . This implies that the action of G on \mathbf{X}' obtained above indeed rectifies the homotopy action $\alpha : G \rightarrow \mathcal{E}_{\mathbf{X}}$. \square

Since $\mathbf{B}\mathcal{E}_{\mathbf{X}} \simeq P^1 \mathbf{B}\mathcal{G}_{\mathbf{X}}$, applying the usual obstruction theory for lifting a map along the Postnikov tower of its target (cf. [Sp, Ch. 8, §2]) yields:

2.12. Corollary ([C, Theorem 1.1(b)]). *Given a homotopy action $\alpha : G \rightarrow \mathcal{E}_{\mathbf{X}}$, there is a sequence of obstructions in $H^{n+2}(G; \pi_{n+1} \mathcal{G}_{\mathbf{X}})$ ($n \geq 1$) to realizing α .*

Compare [SV1, Theorem 3.1] for an obstruction theory in terms of homotopy-coherent actions.

There is also a sequence of difference obstructions in $H^{n+1}(G; \pi_{n+1} \mathcal{G}_{\mathbf{X}})$ for distinguishing between non-homotopic lifts.

3. REALIZING POINTED HOMOTOPY ACTIONS

Cooke's approach to homotopic lifts determines the resulting G -space up to Borel weakly equivalence, since the G -space we obtain has a free G -action. There is also a pointed version:

3.1. Definition. If $\mathbf{X}_* = (\mathbf{X}, x_0)$ is a pointed G -space (§0.1) and the G -action is free on $\mathbf{X} \setminus \{x_0\}$, we call \mathbf{X}_* a *free pointed G -space*. For any pointed G -space, the *associated free pointed G -space* is the quotient

$$\mathbf{E}G \ltimes \mathbf{X} := \mathbf{E}G \times \mathbf{X} / \mathbf{E}G \times \{x_0\},$$

with G -action induced from the diagonal action on $\mathbf{E}G \times \mathbf{X}$ (see §1.5). The *pointed Borel construction* on \mathbf{X} is $\mathbf{X}_{hG}^* := \mathbf{E}G \ltimes_G \mathbf{X} = \mathbf{E}G \ltimes \mathbf{X} / G$.

A *homotopy fixed point* for a G -space \mathbf{X} is a G -map $f : \mathbf{E}G \rightarrow \mathbf{X}$.

3.2. Lemma. Any pointed G -space $\mathbf{X}_* = (\mathbf{X}, x_0)$ has a G -map $r : \mathbf{E}G \ltimes \mathbf{X} \rightarrow \mathbf{X}$ which is a pointed homotopy equivalence; if \mathbf{X}_* is a free pointed G -space, the map r is a G -homotopy equivalence.

Proof. We have a diagram of G -spaces:

$$(3.3) \quad \begin{array}{ccccc} & & \varphi & & \\ & \swarrow & & \searrow & \\ \mathbf{E}G \times \{x_0\} & \xrightarrow{j} & \mathbf{E}G \times \mathbf{X} & \xrightarrow{s} & \mathbf{E}G \ltimes \mathbf{X} \\ \downarrow \text{h.e. } p & & \downarrow \text{h.e. } q & & \downarrow r \\ \{x_0\} & \xrightarrow{\quad} & \mathbf{X} & \xrightarrow{\text{Id}} & \mathbf{X} \end{array}$$

where the vertical maps are projections onto the second factor. Since each row is a cofibration sequence and p and q are Borel G -equivalences, so is r . If the pointed action on \mathbf{X}_* is free, r induces homotopy equivalences on all fixed point sets (which consist only of the basepoint for all $\{e\} \neq H \leq G$), so by [JS, Theorem (1.1)] r is in fact a G -homotopy equivalence. \square

3.4. Definition. Recall that a pointed G -space $\mathbf{Y}_* = (\mathbf{Y}, y_0)$ has an action map $\varphi_{\mathbf{Y}_*}^* : G \rightarrow \mathcal{A}_{\mathbf{Y}_*}^*$, with the monoid action map $\zeta : G \rightarrow \mathcal{G}_{\mathbf{Y}_*}^*$. We say that φ^* *realizes* the pointed homotopy action $\alpha^* : G \rightarrow \mathcal{E}_{\mathbf{X}_*}^* := \pi_0 \mathcal{G}_{\mathbf{X}_*}^*$ if we have a pointed homotopy equivalence $h : \mathbf{Y} \rightarrow \mathbf{X}$ such that the diagram of H -space maps of associative (topological) monoids:

$$(3.5) \quad \begin{array}{ccccc} & & \mathcal{A}_{\mathbf{Y}_*}^* & \hookrightarrow & \mathcal{G}_{\mathbf{Y}_*}^* & \xrightarrow{h_*} & \mathcal{G}_{\mathbf{X}_*}^* \\ & \nearrow \varphi^* & & & & & \downarrow \gamma^* \\ G & \xrightarrow{\alpha^*} & & & & & \mathcal{E}_{\mathbf{X}_*}^* \end{array}$$

commutes.

3.6. Definition. A G -action $\varphi : G \rightarrow \mathcal{A}_{\mathbf{X}}$ on a space \mathbf{X} *lifts weakly to a pointed action* $\varphi^* : G \rightarrow \mathcal{A}_{\mathbf{Y}_*}^*$ if we have Borel G -equivalences $p : \mathbf{Z} \rightarrow \mathbf{X}$ and $p' : \mathbf{Z} \rightarrow \mathbf{Y}$, with sections $i : \mathbf{X} \rightarrow \mathbf{Z}$ and $i' : \mathbf{Y} \rightarrow \mathbf{Z}$ as in Lemma 1.7, such that the diagram

of associative topological monoids (and multiplicative maps):

$$(3.7) \quad \begin{array}{ccccccc} & & \mathcal{A}_{\mathbf{Y}_*}^* & \hookrightarrow & \mathcal{G}_{\mathbf{Y}_*}^* & \hookrightarrow & \mathcal{G}_{\mathbf{Y}} & \xrightarrow{i'_*} & \mathcal{G}_{\mathbf{Z}} \\ & \nearrow \varphi^* & & & & & & & \uparrow i_* \\ G & \xrightarrow{\varphi} & \mathcal{A}_{\mathbf{X}} & \hookrightarrow & & \xrightarrow{\quad} & \mathcal{G}_{\mathbf{X}} & & \end{array}$$

commutes up to homotopy after inverting the homotopy equivalence i_* .

3.8. *Remark.* Since we assumed that \mathbf{X} , \mathbf{Y} , and \mathbf{Z} are CW complexes, any homotopy equivalence between them can be made into a pointed homotopy equivalence by choosing appropriate (non-degenerate) base-points (cf. [D, Theorem 3.6]). As a result, we may assume that \mathbf{X} and \mathbf{Y} in Definition 3.6 are pointed, that $p : \mathbf{Z} \rightarrow \mathbf{X}$ and $p' : \mathbf{Z} \rightarrow \mathbf{Y}$ are pointed Borel G -equivalences, with sections $i : \mathbf{X} \rightarrow \mathbf{Z}$ and $i' : \mathbf{Y} \rightarrow \mathbf{Z}$ as in Lemma 1.7, and (3.7) is replaced by

$$(3.9) \quad \begin{array}{ccccccc} & & \mathcal{A}_{\mathbf{Y}_*}^* & \hookrightarrow & \mathcal{G}_{\mathbf{Y}_*}^* & \xrightarrow{i'_*} & \mathcal{G}_{\mathbf{Z}_*}^* & \xleftarrow{i_*} & \mathcal{G}_{\mathbf{X}_*}^* \\ & \nearrow \varphi^* & & & & & & & \downarrow j \\ G & \xrightarrow{\varphi} & \mathcal{A}_{\mathbf{X}} & \hookrightarrow & & \xrightarrow{\quad} & \mathcal{G}_{\mathbf{X}} & & \end{array}$$

We thus see that Definition 3.4 is equivalent (by Remark 1.9 to saying that we have a pointed homotopy equivalence $h : \mathbf{Y}_* \rightarrow \mathbf{X}_*$ such that $h_* \circ \zeta_{\mathbf{Y}}^* \sim \zeta_{\mathbf{X}}^* : G \rightarrow \mathcal{G}_{\mathbf{X}_*}^*$ are homotopic as H -maps.

3.10. **Lemma.** *A G -space \mathbf{X} with action $\varphi : G \rightarrow \mathcal{A}_{\mathbf{X}}$ has a homotopy fixed point corresponding to each lift up to homotopy of φ to a pointed action φ^* .*

Proof. A lift up to homotopy of φ to a pointed action $\varphi^* : G \rightarrow \mathcal{A}_{\mathbf{Y}_*}^*$ yields a fixed point $y_0 \in \mathbf{Y}$, and thus a homotopy fixed point for $\mathbf{Y}_* = (\mathbf{Y}, y_0)$ given by the constant map $c_{y_0} : \mathbf{E}G \rightarrow \mathbf{Y}$ (which is a G -map). This lifts to a homotopy fixed point $\hat{f} : (\text{Id}, c_{y_0}) : \mathbf{E}G \rightarrow \mathbf{E}G \times \mathbf{Y}$. Since $\text{Id} \times h : \mathbf{E}G \times \mathbf{Z} \rightarrow \mathbf{E}G \times \mathbf{Y}$ is a G -map of free G -CW complexes which is also a homotopy equivalence, it is actually a G -homotopy equivalence by [JS, Theorem (1.1)], with G -inverse $h^{-1} : \mathbf{E}G \times \mathbf{Y} \rightarrow \mathbf{E}G \times \mathbf{Z}$. The G -map $k \circ h^{-1} \circ \hat{f} : \mathbf{E}G \rightarrow \mathbf{X}$ is the corresponding homotopy fixed point for \mathbf{X} .

Conversely, if $f : \mathbf{E}G \rightarrow \mathbf{X}$ is a G -map, we may factor f in the model category $G\mathcal{T}$ (see §1.1) as a G -cofibration followed by a G -fibration weak equivalence: $\mathbf{E}G \xrightarrow{\tilde{f}} \mathbf{Z} \xrightarrow{p} \mathbf{X}$. If we let $\mathbf{Y} := \mathbf{Z}/\mathbf{E}G$ denote the (homotopy) cofiber of \tilde{f} , with quotient G -map $p' : \mathbf{Z} \rightarrow \mathbf{Y}$, then \mathbf{Y} has a basepoint y_0 (corresponding to $\mathbf{E}G \subseteq \mathbf{Z}$), fixed under the G -action, and p' is a Borel G -equivalence since $\mathbf{E}G$ is contractible. Thus the G -action on $\mathbf{Y}_* = (\mathbf{Y}, y_0)$ yields the required pointed lift (3.7). \square

By construction a G -action on a space \mathbf{X} is determined up to Borel weak equivalence by a map $\Phi : \mathbf{B}G \rightarrow \mathbf{B}\mathcal{G}_{\mathbf{X}}$, from which we obtain the free G -space $\mathbf{X}_1 \simeq \mathbf{X}$ using (2.7).

3.11. **Proposition.** *Given a pointed space $\mathbf{X}_* = (\mathbf{X}, x_0)$, a G -action on \mathbf{X} corresponding to $\Phi : \mathbf{B}G \rightarrow \mathbf{B}\mathcal{G}_{\mathbf{X}}$ lifts weakly to a pointed free action on \mathbf{X}_* if and*

only if Φ lifts up to homotopy to a map Φ^* in:

$$(3.12) \quad \begin{array}{ccc} & & \mathbf{BG}_{\mathbf{X}^*}^* \\ & \nearrow \Phi^* & \downarrow \mathbf{B}j \\ \mathbf{BG} & \xrightarrow{\Phi} & \mathbf{BG}_{\mathbf{X}} \end{array}$$

Proof. One direction follows from the definition, since applying the classifying space functor \mathbf{B} to (3.9) yields a homotopy commutative diagram of the form (3.12).

Conversely, given (3.12), pulling back the universal fibration $\mathbf{B}j$ of (2.6) along Φ yields the following (homotopy) pullback square:

$$(3.13) \quad \begin{array}{ccc} \mathbf{BG} & \xrightarrow{\Phi^*} & \mathbf{BG}_{\mathbf{X}^*}^* \\ \searrow \sigma & & \downarrow \mathbf{B}j \\ & E_\theta & \xrightarrow{\text{PB}} \\ \downarrow \theta & & \downarrow \mathbf{B}j \\ \mathbf{BG} & \xrightarrow{\Phi} & \mathbf{BG}_{\mathbf{X}} \end{array}$$

=

so we can use Φ^* to obtain a homotopy section $\sigma : \mathbf{BG} \rightarrow E_\theta$ as indicated.

We now use the lower left homotopy pullback square in (2.7) to obtain a homotopy fixed point $f : \mathbf{EG} \rightarrow \mathbf{X}_1$:

$$(3.14) \quad \begin{array}{ccc} \mathbf{EG} & \xrightarrow{=} & \mathbf{EG} \\ \searrow f & & \downarrow q \\ & \mathbf{X}_1 & \xrightarrow{\text{PB}} \\ \downarrow \sigma \circ q & & \downarrow \theta \\ & E_\theta & \xrightarrow{\theta} \mathbf{BG} \end{array}$$

where $\mathbf{X}_1 \simeq \mathbf{X}$. Hence by Lemma 3.10 we obtain a pointed G -action on a pointed space \mathbf{Y}_* homotopy equivalent to \mathbf{X} .

To see that this pointed action realizes Φ^* , as in the proof of Proposition 2.5 we can replace $\Omega \mathbf{BG} \xrightarrow{\Omega \Phi^*} \Omega \mathbf{BG}_{\mathbf{X}^*}^* \xrightarrow{\Omega \mathbf{B}j} \Omega \mathbf{BG}_{\mathbf{X}}$ by injective homomorphisms of topological groups $\widehat{G} \xrightarrow{\widehat{\zeta}^*} \widehat{\mathcal{G}}_{\mathbf{X}}^* \xrightarrow{\widehat{j}} \widehat{\mathcal{G}}_{\mathbf{X}}$. These induce maps of principle bundles:

$$(3.15) \quad \begin{array}{ccccccc} \widehat{G} & \xrightarrow{\widehat{\zeta}^*} & \widehat{\mathcal{G}}_{\mathbf{X}}^* & \xrightarrow{\widehat{j}} & \widehat{\mathcal{G}}_{\mathbf{X}} & \xleftarrow{=} & \widehat{\mathcal{G}}_{\mathbf{X}} \\ \downarrow & & \downarrow & \searrow \widehat{j} & \downarrow & & \downarrow \\ \mathbf{E}\widehat{G} & \xrightarrow{\mathbf{E}\widehat{\zeta}^*} & \mathbf{E}\widehat{\mathcal{G}}_{\mathbf{X}}^* & \xrightarrow{k} & \mathbf{E}\widehat{\mathcal{G}}_{\mathbf{X}} & \xleftarrow{=} & \mathbf{E}\widehat{\mathcal{G}}_{\mathbf{X}} \\ \downarrow & & \downarrow & \searrow \mathbf{E}\widehat{j} & \downarrow & & \downarrow \\ \mathbf{B}\widehat{G} & \xrightarrow{\mathbf{B}\widehat{\zeta}^*} & \mathbf{B}\widehat{\mathcal{G}}_{\mathbf{X}}^* & \xrightarrow{\sim} & \mathbf{B}\widehat{\mathcal{G}}_{\mathbf{X}} & \xleftarrow{\widetilde{\mathbf{B}j}} & \widetilde{\mathbf{B}\mathcal{G}}_{\mathbf{X}}^* \\ & & & & & & \downarrow \eta \\ & & & & & & \mathbf{X}'' \end{array}$$

using the notations of (2.10), where the induced map of total spaces $k : \mathbf{E}\widehat{\mathcal{G}}_{\mathbf{X}}^* \rightarrow \widehat{\mathbf{X}}''$ is defined by the pullback property.

Note that because \widehat{j} is an inclusion, so is $\mathbf{E}\widehat{j}$, and thus k . The latter is the *universal homotopy fixed point* for $\widehat{\mathbf{X}}''$, in the following sense: if we apply the construction of Lemma 3.10 to obtain a cofibration sequence $\mathbf{E}\widehat{\mathcal{G}}_{\mathbf{X}}^* \xrightarrow{k} \widehat{\mathbf{X}}'' \xrightarrow{q} \mathbf{X}'_*$ of $\widehat{\mathcal{G}}_{\mathbf{X}}^*$ -maps (so q is a homotopy equivalence and \mathbf{X}'_* is free pointed $\widehat{\mathcal{G}}_{\mathbf{X}}^*$ -space), then the composite $\zeta_{\mathbf{X}'_*}^*$ of $\widehat{\mathcal{G}}_{\mathbf{X}}^* \xrightarrow{\varphi_{\mathbf{X}'_*}^*} \mathcal{A}_{\mathbf{X}'_*}^* \hookrightarrow \mathcal{G}_{\mathbf{X}'_*}^*$ is a homotopy equivalence.

Note that the composite \widehat{G} -homotopy fixed point $k \circ \mathbf{E}\widehat{\zeta}^* : \mathbf{E}\widehat{G} \rightarrow \widehat{\mathbf{X}}''$ is, up to homotopy, the G -homotopy fixed point $f : \mathbf{E}G \rightarrow \mathbf{X}_1$: to see this, note that the corresponding map $\widehat{f} : \mathbf{E}\widehat{G} \rightarrow \widehat{\mathbf{X}}''$ is in fact part of the (unique) dotted lift of vertical bundle maps:

$$(3.16) \quad \begin{array}{ccccc} \widehat{G} & & & & \\ \downarrow & \searrow \widehat{\zeta} & & \searrow & \\ \mathbf{E}\widehat{G} & & \widehat{\mathcal{G}}_{\mathbf{X}} & \xleftarrow{\widehat{\zeta}} & \widehat{G} \\ \downarrow & \searrow k \circ \mathbf{E}\widehat{\zeta}^* & \downarrow & & \downarrow \\ \mathbf{B}\widehat{G} & & \widehat{\mathbf{X}}'' & \xleftarrow{\simeq} & \widehat{\mathbf{X}}' \\ & \searrow \widehat{\sigma} & \downarrow \eta & & \downarrow \widehat{\xi} \\ & \mathbf{B}\widehat{\zeta}^* & \widehat{\mathbf{B}\mathcal{G}}_{\mathbf{X}}^* & \xleftarrow{q} & E_{\widehat{\theta}} \end{array}$$

(again using the notations of (2.10)). This shows that the map $k \circ \mathbf{E}\widehat{\zeta}^*$, as a map of free \widehat{G} -spaces, is classified by the splitting $\widehat{\sigma} : \mathbf{B}\widehat{G} \rightarrow E_{\widehat{\theta}}$, which up to homotopy is the same as $\sigma : \mathbf{B}G \rightarrow E_{\theta}$ in (3.13), classifying the homotopy fixed point f of (3.14).

As in the proof of Proposition 2.5, the G -action on $\widehat{\mathbf{X}}'$ corresponds under the homotopy equivalence $h : \widehat{\mathbf{X}}' \rightarrow \widehat{\mathbf{X}}''$ to the $\widehat{\mathcal{G}}_{\mathbf{X}}$ -action on $\widehat{\mathbf{X}}''$ via the homomorphism of topological groups $\widehat{\zeta} : \widehat{G} \rightarrow \widehat{\mathcal{G}}_{\mathbf{X}}$. Using Lemma 3.10 and the homotopy fixed point $f : \mathbf{E}G \rightarrow \mathbf{X}_1$, we obtained a free pointed G -action on $\mathbf{X}_{1*} \simeq \mathbf{X}_1$. Up to homotopy, this is the same as the free pointed \widehat{G} -action on the pointed space $\mathbf{X}'_* := \widehat{\mathbf{X}}''/\mathbf{E}\widehat{\mathcal{G}}_{\mathbf{X}}^*$, which was induced from the universal free pointed $\widehat{\mathcal{G}}_{\mathbf{X}}^*$ -action on \mathbf{X}'_* by the homomorphism $\widehat{\zeta}^* : \widehat{G} \rightarrow \widehat{\mathcal{G}}_{\mathbf{X}}^*$. Since the action map $\zeta_{\mathbf{X}'_*}^* : \widehat{\mathcal{G}}_{\mathbf{X}}^* \rightarrow \mathcal{A}_{\mathbf{X}'_*}^* \hookrightarrow \mathcal{G}_{\mathbf{X}'_*}^*$ is a homotopy equivalence, the corresponding monoid action map $\widehat{G} \rightarrow \mathcal{G}_{\mathbf{X}'_*}^*$ for the \widehat{G} -action on \mathbf{X}'_* is, up to homotopy, just $\widehat{\zeta}^*$. However, $\mathbf{B}\widehat{\zeta}^* : \mathbf{B}\widehat{G} \rightarrow \mathbf{B}\widehat{\mathcal{G}}_{\mathbf{X}}^*$ is, up to homotopy, just the given map $\Phi^* : \mathbf{B}G \rightarrow \mathbf{B}\mathcal{G}_{\mathbf{X}}^*$. \square

3.17. Corollary. *A pointed homotopy action $\alpha^* : G \rightarrow \mathcal{E}_{\mathbf{X}_*}^*$ of G on \mathbf{X}_* can be realized by a pointed (free) G -action if and only if $\mathbf{B}\alpha^* : \mathbf{B}G \rightarrow \mathbf{B}\mathcal{E}_{\mathbf{X}_*}^*$ lifts up to homotopy in (0.3) to a map $\Phi^* : \mathbf{B}G \rightarrow \mathbf{B}\mathcal{G}_{\mathbf{X}_*}^*$.*

Proof. Given a realization φ^* of α^* , we obtain a lifting Φ^* as required by applying the classifying space functor \mathbf{B} to (3.5), using Remark 1.9.

Conversely, given any map $\Phi^* : \mathbf{B}G \rightarrow \mathbf{B}\mathcal{G}_{\mathbf{X}_*}^*$, we can compose it with $\mathbf{B}j : \mathbf{B}\mathcal{G}_{\mathbf{X}_*}^* \rightarrow \mathbf{B}\mathcal{G}_{\mathbf{X}}$ to obtain a map $\Phi : \mathbf{B}G \rightarrow \mathbf{B}\mathcal{G}_{\mathbf{X}}$, which classifies a free G -action

on $\mathbf{X}_1 \simeq \mathbf{X}$ (rectifying the homotopy action $\alpha := \pi_1 \Phi : G \rightarrow \mathcal{E}_{\mathbf{X}}$), by Proposition 2.5. We may now apply Proposition 3.11 to Φ . \square

3.18. Definition. Given a pointed homotopy action $\alpha^* : G \rightarrow \mathcal{E}_{\mathbf{X}_*}^*$ of G on \mathbf{X}_* , let $k_n : P^n \mathbf{B}\mathcal{G}_{\mathbf{X}_*}^* \rightarrow K(\pi_{n+1}(\mathbf{B}\mathcal{G}_{\mathbf{X}_*}^*), n+2)$ denote the n -th k -invariant for the connected space $\mathbf{B}\mathcal{G}_{\mathbf{X}_*}^*$. Assume by induction on $n \geq 1$ that we have constructed a lift $h_n : \mathbf{B}G \rightarrow P^n \mathbf{B}\mathcal{G}_{\mathbf{X}_*}^*$ for $\mathbf{B}\alpha^* : \mathbf{B}G \rightarrow \mathbf{B}\mathcal{E}_{\mathbf{X}_*}^*$. Then the n -th obstruction class for this lift is

$$[k_n \circ h_n] \in [\mathbf{B}G, K(\pi_{n+1}(\mathbf{B}\mathcal{G}_{\mathbf{X}_*}^*), n+2)] \cong H^{n+2}(G; \pi_{n+1}(\mathbf{B}\mathcal{G}_{\mathbf{X}_*}^*)).$$

From Corollary 3.17 we deduce (compare [C, Theorem 1.1(b)]):

3.19. Proposition. A pointed homotopy action $\alpha^* : G \rightarrow \mathcal{E}_{\mathbf{X}_*}^*$ of G on \mathbf{X}_* can be realized by a pointed G -action on \mathbf{X}_* if and only if the obstruction classes $[k_n \circ h_n] \in H^{n+2}(G; \pi_{n+1}(\mathbf{B}\mathcal{G}_{\mathbf{X}_*}^*))$ successively vanish, for some sequence of lifts. There is also a sequence of difference obstructions in $H^{n+1}(G; \pi_{n+1}(\mathbf{B}\mathcal{G}_{\mathbf{X}_*}^*))$ for distinguishing between non-homotopic lifts in (3.12).

4. TRANSFERRING GROUP ACTIONS

We now address the more general problem of extending a given G -action on \mathbf{X} , or lifting a G -action on \mathbf{Y} , with respect to a map $f : \mathbf{X} \rightarrow \mathbf{Y}$. We also discuss the problem of making f equivariant with respect to given actions on both \mathbf{X} and \mathbf{Y} . The results of this section are not actually needed to prove our main theorem, but they may be useful in further applications (and their proofs are used in the next section).

4.1. Definition. Given any map $f : \mathbf{X} \rightarrow \mathbf{Y}$, a G -map $f' : \mathbf{X}' \rightarrow \mathbf{Y}'$ is:

- (i) a *right transfer* of a G -action on \mathbf{X} along f if we have a homotopy-commutative diagram

$$(4.2) \quad \begin{array}{ccc} \mathbf{X} & \xrightarrow{f} & \mathbf{Y} \\ k \uparrow \simeq & & \simeq \downarrow h \\ \mathbf{X}' & \xrightarrow{f'} & \mathbf{Y}' \end{array}$$

in which h is a homotopy equivalence, and k is a Borel G -equivalence.

- (ii) a *left transfer* of a G -action on \mathbf{Y} along f if we have a diagram

$$(4.3) \quad \begin{array}{ccccc} \mathbf{X} & \xrightarrow{f} & \mathbf{Y} & \xleftarrow{m} & \mathbf{Y}'' \\ k \uparrow \simeq & & & \simeq \swarrow & \\ \mathbf{X}' & \xrightarrow{f'} & \mathbf{Y}' & \xleftarrow{n} & \end{array}$$

in which k is a homotopy equivalence, m and n are Borel G -equivalences, which becomes homotopy-commutative after inverting m or n (up to homotopy).

- (iii) a *compatible G -map* for f with respect to G -actions on \mathbf{X} and \mathbf{Y} if we have a diagram

$$(4.4) \quad \begin{array}{ccccc} \mathbf{X} & \xrightarrow{f} & \mathbf{Y} & \xleftarrow{m} & \mathbf{Y}'' \\ k \uparrow \simeq & & & \simeq \swarrow & \\ \mathbf{X}' & \xrightarrow{f'} & \mathbf{Y}' & \xleftarrow{n} & \end{array}$$

in which k , m , and n are all Borel G -equivalences, which becomes homotopy-commutative after inverting m or n .

In order to describe the conditions under which such transfers exist, we require also the following construction:

4.5. Definition. For any map $f : \mathbf{X} \rightarrow \mathbf{Y}$, let \mathcal{P}_f denote the homotopy pullback:

$$(4.6) \quad \begin{array}{ccc} \mathcal{P}_f & \xrightarrow{\varepsilon} & \mathcal{G}_{\mathbf{Y}} \\ \downarrow \delta & \boxed{\text{PB}} & \downarrow f^* \\ \mathcal{G}_{\mathbf{X}} & \xrightarrow{f_*} & \text{Map}(\mathbf{X}, \mathbf{Y}) . \end{array}$$

This can be constructed explicitly in two ways: if we change f into a cofibration, the map $f^* : \text{Map}(\mathbf{Y}, \mathbf{Y}) \rightarrow \text{Map}(\mathbf{X}, \mathbf{Y})$ is a fibration, so its restriction to $\mathcal{G}_{\mathbf{Y}}$ is a fibration, too (since the latter is just a union of path components of $\text{Map}(\mathbf{Y}, \mathbf{Y})$). In this case, the strict pullback is actually the homotopy pullback. Similarly when f is a fibration, so f_* is a fibration.

Using such a strict model, we see that \mathcal{P}_f is a sub-monoid of the strictly associative monoid $\mathcal{G}_{\mathbf{X}} \times \mathcal{G}_{\mathbf{Y}}$. Moreover, it is grouplike, since $(g, h) \in \mathcal{P}_f$ means that $f \circ g = h \circ f$ (for $g \in \mathcal{G}_{\mathbf{X}}$ and $h \in \mathcal{G}_{\mathbf{Y}}$), and thus $f \circ g^{-1} \sim h^{-1} \circ f$. If f is either a fibration or a cofibration, we can use [BJT, Lemma 4.16] to change h^{-1} (respectively, g^{-1}) up to homotopy to get $f \circ g^{-1} = h^{-1} \circ f$ and thus $(g^{-1}, h^{-1}) \in \mathcal{P}_f$, too. The maps δ and ε (the restrictions of the structure maps for $\tilde{\mathcal{P}}_f$) are monoid maps. Evidently \mathcal{P}_f is a homotopy invariant of f .

With these notions we then have the following:

4.7. Proposition. Let $f : \mathbf{X} \rightarrow \mathbf{Y}$ be any map in \mathcal{T} .

- (i) There is a right transfer of a G -action on \mathbf{X} (with monoid action map $\zeta_{\mathbf{X}} : G \rightarrow \mathcal{G}_{\mathbf{X}}$) along f if and only if there is a map Ψ making the following diagram commute up to homotopy:

$$(4.8) \quad \begin{array}{ccc} & & \mathbf{B}\mathcal{P}_f \\ & \nearrow \Psi & \downarrow \mathbf{B}\delta \\ \mathbf{B}G & \xrightarrow{\mathbf{B}\zeta_{\mathbf{X}}} & \mathbf{B}\mathcal{G}_{\mathbf{X}} . \end{array}$$

- (ii) There is a left transfer of a G -action on \mathbf{Y} (with monoid action map $\zeta_{\mathbf{Y}} : G \rightarrow \mathcal{G}_{\mathbf{Y}}$) along f if and only if there is a map Ψ making the following diagram commute up to homotopy:

$$(4.9) \quad \begin{array}{ccc} & & \mathbf{B}\mathcal{P}_f \\ & \nearrow \Psi & \downarrow \mathbf{B}\varepsilon \\ \mathbf{B}G & \xrightarrow{\mathbf{B}\zeta_{\mathbf{Y}}} & \mathbf{B}\mathcal{G}_{\mathbf{Y}} . \end{array}$$

- (iii) There is a compatible G -map for f with respect to G -actions on \mathbf{X} and \mathbf{Y} (with monoid action maps $\zeta_{\mathbf{X}} : G \rightarrow \mathcal{G}_{\mathbf{X}}$ and $\zeta_{\mathbf{Y}} : G \rightarrow \mathcal{G}_{\mathbf{Y}}$) if and only if

there is a map Ψ making the following diagram commute up to homotopy:

$$(4.10) \quad \begin{array}{ccc} & & \mathbf{BP}_f \\ & \nearrow \Psi & \downarrow (\mathbf{B}\delta, \mathbf{B}\varepsilon) \\ \mathbf{BG} & \xrightarrow{(\mathbf{B}\zeta_X, \mathbf{B}\zeta_Y)} & \mathbf{BG}_X \times \mathbf{BG}_Y. \end{array}$$

Proof. (i) If \mathbf{X} is a G -space, and we have a right transfer $f' : \mathbf{X}' \rightarrow \mathbf{Y}'$, of the G -action along f , we may change f' into a G -cofibration, and the monoid action maps $\zeta_{\mathbf{X}'} : G \rightarrow \mathcal{G}_{\mathbf{X}'}$ and $\zeta_{\mathbf{Y}'} : G \rightarrow \mathcal{G}_{\mathbf{Y}'}$ then fit together to define a monoid map $z : G \rightarrow \tilde{\mathcal{P}}_{f'}$ in (4.6), which actually lands in $\mathcal{P}_{f'}$, since G is a group. Because $\mathbf{B}\zeta_{\mathbf{X}}$ is just $\mathbf{B}\zeta_{\mathbf{X}'}$, up to homotopy, $\mathbf{B}z : \mathbf{BG} \rightarrow \mathbf{BP}_{f'}$ is the required lift in (4.8).

Conversely, given a lift Ψ in (4.8), as in the proof of Proposition 2.5, by applying Kan's G -functor to (4.8), and taking cofibrant replacement in the model category of strictly associative topological monoids (see [SV2, Theorem B]), we obtain a diagram of cofibrant (and grouplike) topological monoids:

$$(4.11) \quad \begin{array}{ccccc} & & \widehat{\mathcal{P}}_f & & \\ & \nearrow \widehat{\rho} & \downarrow \widehat{\delta} & \searrow \widehat{\varepsilon} & \\ \widehat{G} & & \widehat{\mathcal{G}}_X & & \widehat{\mathcal{G}}_Y, \end{array}$$

with $\widehat{\mathcal{P}}_f$ weakly equivalent to \mathcal{P}_f .

Since $\mathbf{BG}_{\widehat{\mathbf{X}}} \simeq \mathbf{BG}_X$, by pulling back (2.6) we obtain a monoid action of \widehat{G} on $\widehat{\mathbf{X}} \simeq \mathbf{X}$. By [Pr, Theorem 5.8] (see also [DL, F2], [M2, §7,9], [Bo, §5] and [DDK]), this is classified by a map $\hat{\theta} : E_{\hat{\theta}} \rightarrow \mathbf{B}\widehat{G}$. Up to homotopy, $\hat{\theta}$ corresponds to the map θ in the fibre bundle sequence:

$$(4.12) \quad \mathbf{X}' := \mathbf{EG} \times \mathbf{X} \rightarrow E_{\theta} \xrightarrow{\theta} \mathbf{BG}$$

classifying the free G -action on \mathbf{X}' (Borel equivalent to the given \mathbf{X}). Similarly, we get a free \widehat{G} -action on $\widehat{\mathbf{Y}} \simeq \mathbf{Y}$, classified by $\hat{\kappa} : E_{\hat{\kappa}} \rightarrow \mathbf{B}\widehat{G}$. Moreover, we have a map $\widehat{f} : \widehat{\mathbf{X}} \rightarrow \widehat{\mathbf{Y}}$ (which is just $f : X \rightarrow Y$, up to homotopy), and we may assume that \widehat{f} is itself a cofibration (for example, by carrying out the above construction in simplicial sets, and replacing $\widehat{\mathbf{Y}}$ by $\widehat{\mathbf{Y}} \times \mathbf{C}\widehat{\mathbf{X}}$ before realizing).

As in (4.6), we obtain a commuting diagram

$$(4.13) \quad \begin{array}{ccc} \mathcal{P}_{\widehat{f}} & \xrightarrow{\widehat{\varepsilon}'} & \mathcal{G}_{\widehat{\mathbf{Y}}} \\ \widehat{\delta}' \downarrow & & \downarrow \widehat{f}^* \\ \mathcal{G}_{\widehat{\mathbf{X}}} & \xrightarrow{\widehat{f}_*} & \text{Map}(\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}), \end{array}$$

in which \widehat{f}^* is a fibration, so $\widehat{\delta}'$ is, too.

Because $\widehat{\mathbf{X}} \simeq \mathbf{X}$, $\widehat{\mathcal{G}}_X$ and \mathcal{G}_X are weakly equivalent, and since the former is cofibrant and the latter is fibrant, we have a weak equivalence of monoids $k : \widehat{\mathcal{G}}_X \rightarrow \mathcal{G}_X$, and similarly $\ell : \widehat{\mathcal{G}}_Y \simeq \mathcal{G}_Y$. Moreover, since (4.13) is a homotopy pullback, \mathcal{P}_f and $\mathcal{P}_{\widehat{f}}$ are weakly equivalent, and again we have a weak equivalence of monoids

$h : \widehat{\mathcal{P}}_f \xrightarrow{\sim} \mathcal{P}_{\widehat{f}}$. Thus the strict diagram (4.13) fits into a homotopy commutative diagram:

$$(4.14) \quad \begin{array}{ccccc} \widehat{\mathcal{P}}_f & \xrightarrow{\widehat{\varepsilon}} & \widehat{\mathcal{G}}_{\mathbf{Y}} & & \\ \downarrow \widehat{\delta} & \searrow h & \downarrow \ell & & \\ \widehat{\mathcal{G}}_{\mathbf{X}} & \xrightarrow{k} & \mathcal{P}_{\widehat{f}} & \xrightarrow{\widehat{\varepsilon}'} & \mathcal{G}_{\widehat{\mathbf{Y}}} \\ & \searrow & \downarrow \widehat{\delta}' & & \downarrow \widehat{f}_* \\ & & \mathcal{G}_{\widehat{\mathbf{X}}} & \xrightarrow{\widehat{f}_*} & \text{Map}(\widehat{\mathbf{X}}, \widehat{\mathbf{Y}}) \end{array}$$

In the model category of strictly associative monoids, we can replace h by another weak equivalence of monoids making $\boxed{\text{A}}$ commute on the nose (cf. [BJT, Lemma 4.16]), and then changing ℓ into a fibration, we may replace $\widehat{\varepsilon}$ by a map making $\boxed{\text{B}}$ commute strictly, too, without changing $\widehat{\mathcal{P}}_f$.

Composing the monoid map $\widehat{\rho} : \widehat{G} \rightarrow \widehat{\mathcal{P}}_f$ of (4.11) with $k \circ \widehat{\delta} : \widehat{\mathcal{P}}_f \rightarrow \mathcal{G}_{\widehat{\mathbf{X}}}$ and $\ell \circ \widehat{\varepsilon} : \widehat{\mathcal{P}}_f \rightarrow \mathcal{G}_{\widehat{\mathbf{Y}}}$, we obtain monoid action maps $\widehat{\zeta}_{\widehat{\mathbf{X}}} : \widehat{G} \rightarrow \mathcal{G}_{\widehat{\mathbf{X}}}$ and $\widehat{\zeta}_{\widehat{\mathbf{Y}}} : \widehat{G} \rightarrow \mathcal{G}_{\widehat{\mathbf{Y}}}$ making $\widehat{\mathbf{X}}$ and $\widehat{\mathbf{Y}}$ into strict \widehat{G} -spaces, with $\widehat{f} : \widehat{\mathbf{X}} \hookrightarrow \widehat{\mathbf{Y}}$ a \widehat{G} -map (which is a cofibration). It therefore fits into a commuting diagram of principal \widehat{G} -bundles

$$(4.15) \quad \begin{array}{ccc} \widehat{\mathbf{X}} & \xrightarrow{\widehat{f}} & \widehat{\mathbf{Y}} \\ \downarrow & & \downarrow \\ E_{\widehat{\theta}} & \xrightarrow{\widehat{\beta}} & E_{\widehat{\kappa}} \\ & \searrow \widehat{\theta} & \swarrow \widehat{\kappa} \\ & \mathbf{B}\widehat{G} & \end{array}$$

Here $\widehat{\beta}$ is obtained by realizing the bar construction for the \widehat{G} -actions on $\widehat{\mathbf{X}}$ and $\widehat{\mathbf{Y}}$, respectively (see [Pr, §5]), so it commutes up to homotopy with the classifying maps $\widehat{\theta}$ and $\widehat{\kappa}$ for the two bundles, where (as noted above) up to homotopy $\widehat{\theta}$ is just $\theta : E_{\theta} \rightarrow \mathbf{B}G$, classifying the free G -space \mathbf{X}' .

Let $\kappa : E_{\widehat{\kappa}} \rightarrow \mathbf{B}G$ denote the composite of $\widehat{\kappa}$ with $\mathbf{B}\widehat{G} \simeq \mathbf{B}G$, classifying a free G -bundle $\mathbf{Y}' \rightarrow E_{\widehat{\kappa}}$ with $\mathbf{Y}' \simeq \widehat{\mathbf{Y}} \simeq \mathbf{Y}$. If we also let $\beta : E_{\theta} \rightarrow E_{\widehat{\kappa}}$ denote the composite of $\widehat{\beta}$ with $E_{\theta} \simeq E_{\widehat{\theta}}$, then $\kappa \circ \beta \simeq \theta$, so we have a map

$$(4.16) \quad \begin{array}{ccc} \mathbf{X}' & \xrightarrow{f'} & \mathbf{Y}' \\ \downarrow & & \downarrow \\ E_{\theta} & \xrightarrow{\beta} & E_{\widehat{\kappa}} \\ & \searrow \theta & \swarrow \kappa \\ & \mathbf{B}G & \end{array}$$

of principal G -bundles, so in particular, f' is a G -map.

(ii) and (iii) are proven analogously. \square

Compare [Z, Proposition 2.2] for the compatibility version for homotopy actions.

4.17. Proposition. *Let $f : \mathbf{X} \rightarrow \mathbf{Y}$ be any map. In a right transfer of a G -action on \mathbf{X} along f , we may assume that k in (4.2) is a homeomorphism; in a left transfer of a G -action on \mathbf{Y} along f , we may assume that m and n in (4.3) are homeomorphisms; and in a compatible G -map for f with respect to G -actions on \mathbf{X} and \mathbf{Y} , we may assume that either k , or m and n , are homeomorphisms in (4.4).*

Proof. If the action of G on \mathbf{X} is free (and (4.8) holds), we can replace $\mathbf{X}' := \mathbf{E}G \times \mathbf{X}$ by \mathbf{X} in (4.12), and therefore also in (4.16), so we have a right transfer $f' : \mathbf{X} \hookrightarrow \mathbf{Y}'$ (along $f : \mathbf{X} \rightarrow \mathbf{Y}$) which is a G -map.

The same argument shows that if the action of G on \mathbf{Y} is free (and (4.9) holds), it has a left transfer along f to a fibration $f' : \mathbf{X}' \twoheadrightarrow \mathbf{Y}$ which is a G -map. Moreover, given free G -actions on both \mathbf{X} and \mathbf{Y} , any $f : \mathbf{X} \rightarrow \mathbf{Y}$ has a compatible G -map $f' : \mathbf{X} \rightarrow \mathbf{Y}$ with the same source and target (if (4.10) holds).

Note that every G -space \mathbf{Y} has a Borel G -equivalence $h : \mathbf{Y}' \rightarrow \mathbf{Y}$, where \mathbf{Y}' is a free G -space (§1.5). Therefore, we do not actually need to assume that the action on \mathbf{Y} is free, since we can compose the left transfer or compatible map f' with this h . Thus we may always assume that ℓ , m , and n are homeomorphisms.

Finally, given a G -space \mathbf{X} , we may replace it by the free G -space $\mathbf{X}' := \mathbf{X} \times \mathbf{E}G$ and produce a G -map $f' : \mathbf{X}' \hookrightarrow \mathbf{Y}'$ which is a cofibration. Then taking the pushout:

$$(4.18) \quad \begin{array}{ccc} \mathbf{X}' & \xrightarrow{f'} & \mathbf{Y}' \\ \simeq \downarrow q & \boxed{\text{PO}} & \simeq \downarrow r \\ \mathbf{X} & \xrightarrow{f''} & \mathbf{Y}'' \end{array}$$

we see that all maps are G -maps; f' , and thus f'' , are cofibrations, so this is a homotopy pushout in \mathcal{T} , and since q is a homotopy equivalence, so is r . \square

The analogue of Definition 3.18 allows us to deduce an analogue of Proposition 3.19 in this case, too.

5. INTERPOLATING GROUP ACTIONS

Next, we consider a sequence of maps

$$(5.1) \quad \mathbf{X} \xrightarrow{f} \mathbf{Y} \xrightarrow{g} \mathbf{Z}$$

with given G -actions on \mathbf{X} and \mathbf{Z} , for which we want to “interpolate” a compatible G -action on \mathbf{Y} .

5.2. Definition. A G -interpolation for two G -spaces \mathbf{X} and \mathbf{Z} and maps as in (5.1) is a pair of G -maps $f' : \mathbf{X}' \rightarrow \mathbf{Y}'$ and $g' : \mathbf{Y}' \rightarrow \mathbf{Z}'$ fitting into a diagram

$$(5.3) \quad \begin{array}{ccccccc} & & \mathbf{X} & \xrightarrow{f} & \mathbf{Y} & \xrightarrow{g} & \mathbf{Z} & \xleftarrow{m} & \mathbf{Z}'' \\ & \nearrow k & & & \downarrow h & & & \nwarrow n & \\ \mathbf{X}'' & \xrightarrow[\ell]{\simeq} & \mathbf{X}' & \xrightarrow{f'} & \mathbf{Y}' & \xrightarrow{g'} & \mathbf{Z}' & & \end{array}$$

in which k , ℓ , m , and n are all homotopy equivalences and G -maps, and h is a homotopy equivalence, which becomes homotopy-commutative after inverting m or n (up to homotopy).

5.4. Definition. Given two composable maps as in (5.1), let $\mathcal{Q}_{f,g}$ denote the homotopy pullback in:

$$(5.5) \quad \begin{array}{ccc} \mathcal{Q}_{f,g} & \xrightarrow{\nu_g} & \mathcal{P}_g \\ \mu_f \downarrow & \boxed{\text{PB}} & \downarrow \delta_g \\ \mathcal{P}_f & \xrightarrow{\varepsilon_f} & \mathcal{G}_{\mathbf{Y}} . \end{array}$$

(see §4.5). If f and g are cofibrations, both \mathcal{P}_f and \mathcal{P}_g are actually pullbacks, the map $\delta : \mathcal{P}_g \rightarrow \mathcal{G}_{\mathbf{Y}}$ is a fibration, so $\mathcal{Q}_{f,g}$ is the ordinary pullback. Furthermore, it is a grouplike strictly associative monoid, and the maps μ and ν are monoid maps.

As in Proposition 4.7 we have:

5.6. Proposition. *Two maps as in (5.1) for G -spaces \mathbf{X} and \mathbf{Z} (with monoid action maps $\zeta_{\mathbf{X}} : G \rightarrow \mathcal{G}_{\mathbf{X}}$ and $\zeta_{\mathbf{Z}} : G \rightarrow \mathcal{G}_{\mathbf{Z}}$, respectively) have a G -interpolation if and only if there is a map Ψ making the following diagram commute up to homotopy:*

$$(5.7) \quad \begin{array}{ccccc} & & & & \mathbf{B}\mathcal{G}_{\mathbf{X}} \\ & & \nearrow \mathbf{B}\zeta_{\mathbf{X}} & & \\ \mathbf{B}G & \xrightarrow{\Psi} & \mathbf{B}\mathcal{Q}_{f,g} & \xrightarrow{\mathbf{B}\delta_f \circ \mathbf{B}\mu} & \\ & \searrow \mathbf{B}\zeta_{\mathbf{Z}} & & \searrow \mathbf{B}\varepsilon_g \circ \mathbf{B}\nu & \\ & & & & \mathbf{B}\mathcal{G}_{\mathbf{Z}} . \end{array}$$

Proof. Given a G -interpolation, the diagram (5.7) is obtained by applying \mathbf{B} to the corresponding monoid action maps.

Conversely, as in the proof of Proposition 2.5, a homotopy commutative diagram (5.7) may be lifted (together with (5.5)) to a commuting diagram of topological monoids:

$$(5.8) \quad \begin{array}{ccccc} \widehat{G} & \xrightarrow{\widehat{\rho}} & \widehat{\mathcal{Q}}_{f,g} & & \\ & \searrow \widehat{\mu}_f & \searrow \widehat{\nu}_g & & \\ & \widehat{\mathcal{P}}_f & & \widehat{\mathcal{P}}_g & \\ \widehat{\mathcal{G}}_{\mathbf{X}} & \xleftarrow{\widehat{\delta}_f} \widehat{\mathcal{P}}_f & \xleftarrow{\widehat{\varepsilon}_f} \widehat{\mathcal{G}}_{\mathbf{Y}} & \xleftarrow{\widehat{\delta}_g} \widehat{\mathcal{P}}_g & \xleftarrow{\widehat{\varepsilon}_g} \widehat{\mathcal{G}}_{\mathbf{Z}} \end{array}$$

and we have spaces $\widehat{\mathbf{X}} \simeq \mathbf{X}$, $\widehat{\mathbf{Y}} \simeq \mathbf{Y}$, and $\widehat{\mathbf{Z}} \simeq \mathbf{Z}$ on which $\widehat{\mathcal{G}}_{\mathbf{X}}$, $\widehat{\mathcal{G}}_{\mathbf{Y}}$, and $\widehat{\mathcal{G}}_{\mathbf{Z}}$, respectively act. Moreover, we have maps $\widehat{f} : \widehat{\mathbf{X}} \rightarrow \widehat{\mathbf{Y}}$ and $\widehat{g} : \widehat{\mathbf{Y}} \rightarrow \widehat{\mathbf{Z}}$ (corresponding up to homotopy to f and g , respectively), and as in the proof of Proposition 4.7, we may assume \widehat{f} and \widehat{g} are cofibrations.

Therefore, (5.8) fits into a diagram:

$$(5.9) \quad \begin{array}{ccccc} & & \widehat{\mathcal{Q}}_{f,g} & & \\ & \swarrow \widehat{\mu}_f & & \searrow \widehat{\nu}_g & \\ & \widehat{\mathcal{P}}_f & & \widehat{\mathcal{P}}_g & \\ & \swarrow \widehat{\delta}_f & \searrow \widehat{\varepsilon}_f & \swarrow \widehat{\delta}_g & \searrow \widehat{\varepsilon}_g \\ \widehat{\mathcal{G}}_{\mathbf{X}} & & \widehat{\mathcal{G}}_{\mathbf{Y}} & & \widehat{\mathcal{G}}_{\mathbf{Z}} \\ \downarrow \ell & & \downarrow m & & \downarrow n \\ & \swarrow \delta_{\widehat{f}} & \searrow \varepsilon_{\widehat{f}} & \swarrow \delta_{\widehat{g}} & \searrow \varepsilon_{\widehat{g}} \\ & \mathcal{P}_{\widehat{f}} & & \mathcal{P}_{\widehat{g}} & \\ & \swarrow \delta_{\widehat{f}} & \searrow \varepsilon_{\widehat{f}} & \swarrow \delta_{\widehat{g}} & \searrow \varepsilon_{\widehat{g}} \\ \mathcal{G}_{\widehat{\mathbf{X}}} & & \mathcal{G}_{\widehat{\mathbf{Y}}} & & \mathcal{G}_{\widehat{\mathbf{Z}}} \end{array}$$

in which $\mathcal{P}_{\widehat{f}}$ and $\mathcal{P}_{\widehat{g}}$ are homotopy limits, so the homotopy equivalences ℓ , m , and n induce homotopy equivalences h and k making (5.9) homotopy-commutative. Since $\widehat{\mathcal{Q}}_{f,g}$ is also a homotopy limit, h and k induce a homotopy equivalence $p : \widehat{\mathcal{Q}}_{f,g} \rightarrow \mathcal{Q}_{\widehat{f},\widehat{g}}$.

Composing $p \circ \widehat{\rho} : \widehat{G} \rightarrow \mathcal{Q}_{\widehat{f},\widehat{g}}$ of diagram (5.8) with the appropriate structure maps for $\mathcal{Q}_{\widehat{f},\widehat{g}}$ yields monoid action maps $\widehat{\zeta}_{\widehat{\mathbf{X}}} : \widehat{G} \rightarrow \mathcal{G}_{\widehat{\mathbf{X}}}$, $\widehat{\zeta}_{\widehat{\mathbf{Y}}} : \widehat{G} \rightarrow \mathcal{G}_{\widehat{\mathbf{Y}}}$, and $\widehat{\zeta}_{\widehat{\mathbf{Z}}} : \widehat{G} \rightarrow \mathcal{G}_{\widehat{\mathbf{Z}}}$ making \widehat{f} and \widehat{g} into \widehat{G} -equivariant maps (by definition of $\mathcal{Q}_{\widehat{f},\widehat{g}}$), with $\widehat{\zeta}_{\widehat{\mathbf{X}}}$ and $\widehat{\zeta}_{\widehat{\mathbf{Z}}}$ corresponding up to homotopy to the given monoid action maps $\zeta_{\mathbf{X}} : G \rightarrow \mathcal{G}_{\mathbf{X}}$ and $\zeta_{\mathbf{Z}} : G \rightarrow \mathcal{G}_{\mathbf{Z}}$.

Passing to the classifying maps for the corresponding principle G - and \widehat{G} -bundles as in the proof of Proposition 4.7, we obtain the required G -interpolation. \square

Aside from the obvious analogue of Proposition 4.17, we have:

5.10. Proposition. *If $p := g \circ f : \mathbf{X} \rightarrow \mathbf{Z}$ is a G -map in (5.1), in any G -interpolation for we may assume that k , ℓ , m , and n are homeomorphisms in (5.3), and that $g' \circ f' = p$.*

Proof. In the proof of Proposition 5.6 we saw that the lifting Ψ in (5.7) allows us to factor the map of free G -spaces (i.e., total spaces of principle G -bundles)

$$\mathbf{X}' := \mathbf{X} \times \mathbf{E}G \xrightarrow{p' := p \times \text{Id}_{\mathbf{E}G}} \mathbf{Z} \times \mathbf{E}G =: \mathbf{Z}'$$

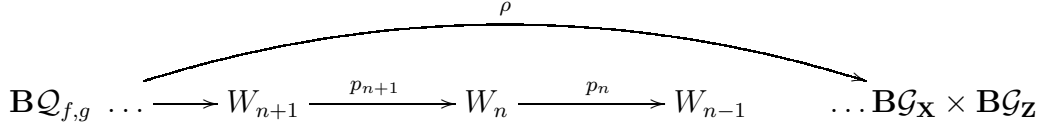
as the composite of two maps of free G -spaces $\mathbf{X}' \xrightarrow{f'} \mathbf{Y}' \xrightarrow{g'} \mathbf{Z}'$ with $\mathbf{Y} \simeq \mathbf{Y}'$ (using a homotopy-factorization of the corresponding classifying maps of the bundles). Applying the (homotopy) pushout (4.18) we obtain a commutative diagram of G -spaces:

$$(5.11) \quad \begin{array}{ccccccc} \mathbf{X}' & \xrightarrow{f'} & \mathbf{Y}' & \xrightarrow{g'} & \mathbf{Z}' & \xrightarrow{q_{\mathbf{Z}}} & \mathbf{Z} \\ \simeq \downarrow q_{\mathbf{X}} & & \downarrow r & & & & \\ \mathbf{X} & \xrightarrow{f''} & \mathbf{Y}'' & \xrightarrow{g''} & \mathbf{Z} & & \end{array}$$

[PO]

5.12. Definition. Given two maps as in (5.1) for G -spaces \mathbf{X} and \mathbf{Z} , let $\rho : \mathbf{B}\mathcal{Q}_{f,g} \rightarrow \mathbf{B}\mathcal{G}_{\mathbf{X}} \times \mathbf{B}\mathcal{G}_{\mathbf{Z}}$ be the map $(\mathbf{B}\delta_f \circ \mathbf{B}\mu, \mathbf{B}\varepsilon_g \circ \mathbf{B}\nu)$ of (5.7), and let:

(5.13)



If F denotes the homotopy fiber of ρ , then up to homotopy each map $p_{n+1} : W_{n+1} \rightarrow W_n$ is a fibration with fiber $K(\pi_{n+1}F, n+1)$, which is classified by a map $\tilde{k}_n : W_n \rightarrow K_{\pi_1 Q_n}(\pi_{n+1}F, n+2)$ (see [R, Theorem 3.4]). Assume by induction on $n \geq 0$ that we have constructed a lift $g_n : \mathbf{B}G \rightarrow W_n$ for $g_0 := (\mathbf{B}\zeta_{\mathbf{X}}, \mathbf{B}\zeta_{\mathbf{Z}}) : \mathbf{B}G \rightarrow W_0 = \mathbf{B}\mathcal{G}_{\mathbf{X}} \times \mathbf{B}\mathcal{G}_{\mathbf{Z}}$. Then the n -th obstruction class for this lift is

$$[\tilde{k}_n \circ g_n] \in [\mathbf{BG}, K_{\pi_1 Q_n}(\pi_{n+1}F, n+2)] \cong H^{n+2}(G; \pi_{n+1}F) ,$$

From Proposition 5.6 we deduce:

6. REALIZING DIAGRAMMATIC HOMOTOPY ACTIONS

6.1. Filtering $\mathcal{O}_G^{\text{op}}$. By [DK, Theorem 3.1], the function complex $\text{Map}_G(\mathbf{X}, \mathbf{Y})$ of G -maps between two G -CW complexes \mathbf{X} and \mathbf{Y} is weakly equivalent to the mapping space $\text{Map}_{\mathcal{T}\text{-}\mathcal{O}_G^{\text{op}}}(\underline{\mathbf{X}}, \underline{\mathbf{Y}})$ between the corresponding $\mathcal{O}_G^{\text{op}}$ -diagrams (cf. (1.4)), at least if $\underline{\mathbf{X}}$ is cofibrant and $\underline{\mathbf{Y}}$ is fibrant. Thus the study of G -maps between G -spaces (up to homotopy) is reduced to the study of mapping spaces of diagrams.

$$(6.2) \quad H = H_0 < H_1 < H_2 < \dots < H_{k-1} < H_k = G.$$
$$(6.3) \quad \mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \mathcal{F}_k \subset \dots \subset \mathcal{O}_G^{\text{op}}$$

by full subcategories, where $\text{Obj } \mathcal{F}_k := \{G/H \in \mathcal{O}_G^{\text{op}} : \text{len}_G H \leq k\}$ (so $\text{Obj } \mathcal{F}_0 = \{G/G\}$).

Since G is finite, the filtration is exhaustive: if $\text{len}_G\{e\} = N$ – that is, the longest possible sequence (6.2) in G has N inclusions of proper subgroups – then $\mathcal{F}_N = \mathcal{O}_G^{\text{op}}$. We let $\widehat{\mathcal{F}}_k$ denote the collection of subgroups $H < G$ such that $G/H \in \mathcal{F}_k$.

Let $\langle H \rangle := \{H^a : a \in G\}$ denote the conjugacy class of a subgroup $H \leq G$: note that if $H \in \widehat{\mathcal{F}}_k$, then $\langle H \rangle \subseteq \widehat{\mathcal{F}}_k$.

6.4. Definition. Let Λ denote the partially ordered set of subgroups of G ; we can think of the opposite category Λ^{op} as a subcategory of \mathcal{O}_G . The full subcategory Λ_H consists of all subgroups K with $H < K \leq G$, and Λ_k is the full subcategory of objects in filtration \mathcal{F}_k .

6.5. Definition. A *Bredon homotopy action* of G $\langle \widetilde{X}, (\alpha_H^*)_{H \leq G} \rangle$ consists of:

- (i) a diagram $\widetilde{X} : \Lambda^{\text{op}} \rightarrow \mathcal{T}$;
- (ii) For each subgroup $H \leq G$, consider the homotopy cofibration sequence:

$$(6.6) \quad \widetilde{\mathbf{X}}_H \xrightarrow{j_H} \widetilde{X}(H) \xrightarrow{q_H} \widetilde{\mathbf{X}}_H^H,$$

where $\widetilde{\mathbf{X}}_H := \text{hocolim}_{\Lambda_H^{\text{op}}} \widetilde{X}(K)$ and j_H is induced by the inclusions $j : H \hookrightarrow K$. Then $\alpha_H^* : W_H \rightarrow \mathcal{E}_{\widetilde{\mathbf{X}}_H^H}$ is a pointed homotopy W_H -action on the homotopy cofiber $\widetilde{\mathbf{X}}_H^H$ of j_H .

6.7. Definition. A cofibrant diagram $\underline{X}_k : \mathcal{F}_k \rightarrow \mathcal{T}$ *realizes* a Bredon homotopy action $\langle \widetilde{X}, (\alpha_H^*)_{H \leq G} \rangle$ in the k -th filtration if:

- (a) The corresponding homotopy diagram $(\gamma \circ \underline{X}_k)|_{\Lambda_k^{\text{op}}} : \Lambda_k^{\text{op}} \rightarrow \text{ho } \mathcal{T}$ is weakly equivalent to $\gamma \circ \widetilde{X}|_{\Lambda_k^{\text{op}}}$, for $\gamma : \mathcal{T} \rightarrow \text{ho } \mathcal{T}$ the quotient functor.
- (b) if for each $H \in \mathcal{F}_k$, we let $\underline{\mathbf{X}}_H^H$ denote the cofiber of $\text{colim}_{K > H} \underline{X}_k(G/K) \rightarrow \underline{X}_k(G/H)$, then the pointed action of W_H on $\underline{\mathbf{X}}_H^H$ realizes the pointed homotopy action α_H^* .

6.8. Example. If \mathbf{X} is a G -CW complex, let \widetilde{X} be the restriction of $\underline{X} : \mathcal{O}_G^{\text{op}} \rightarrow \mathcal{T}$ to the subcategory Λ^{op} . For any $H \leq G$, $\widetilde{\mathbf{X}}_H := \text{hocolim}_{\Lambda_H^{\text{op}}} \widetilde{X}(K)$ is simply $\mathbf{X}_H := \bigcup_{H < K} \mathbf{X}^K$, which is a sub- W_H -complex of \mathbf{X}^H . The quotient $\mathbf{X}_H^H := \mathbf{X}^H / \mathbf{X}_H$ is the cofiber of the inclusion $j_H : \mathbf{X}_H \hookrightarrow \mathbf{X}^H$, which is a free pointed W_H -space (unless $\mathbf{X}_H = \emptyset$), with monoid action map $\zeta_{\mathbf{X}_H^H}^* : W_H \rightarrow \mathcal{G}_{\mathbf{X}_H^H}^*$ and $\alpha_H^* := \gamma^* \circ \zeta_{\mathbf{X}_H^H}^* : W_H \rightarrow \mathcal{E}_{\mathbf{X}_H^H}$. Evidently $\underline{X} : \mathcal{F}_\infty = \mathcal{O}_G^{\text{op}} \rightarrow \mathcal{T}$ realizes the Bredon homotopy action $\langle \underline{X}|_{\Lambda^{\text{op}}}, (\alpha_H^*)_{H \leq G} \rangle$ we have just defined (in all filtrations). In this case, we also say that the G -space \mathbf{X} *realizes* $\langle \underline{X}|_{\Lambda^{\text{op}}}, (\alpha_H^*)_{H \leq G} \rangle$.

6.9. Definition. If $\underline{X}_k : \mathcal{F}_k \rightarrow \mathcal{T}$ realizes a Bredon homotopy action $\langle \widetilde{X}, (\alpha_H^*)_{H \leq G} \rangle$ in the k -th filtration, for each conjugacy class $\langle H \rangle \subseteq \widehat{\mathcal{F}}_{k+1} \setminus \widehat{\mathcal{F}}_k$, choose any representative $H \in \langle H \rangle$. The two $\langle H \rangle$ -sequences of obstructions $(c_n)_{n=1}^\infty$ and $(e_n)_{n=1}^\infty$ to extending \underline{X}_k to \underline{X}_{k+1} are defined as follows:

- (a) For each $n \geq 1$, the n -th obstruction $c_H^n \in H^{n+2}(W_H; \pi_{n+1}(\mathbf{BG}_{\widetilde{\mathbf{X}}_H^H}^*))$ to realizing the pointed homotopy action α_H^* of W_H on $\widetilde{\mathbf{X}}_H^H$ (cf. Proposition 3.19).

- (b) If we produced a vanishing sequence of obstructions $(c_H^n)_{n=1}^\infty$, and thus realized the pointed homotopy action α_H^* of W_H on $\tilde{\mathbf{X}}_H^H$ by a topological pointed action of W_H on a space $\underline{\mathbf{X}}_H^H \simeq \tilde{\mathbf{X}}_H^H$, which fits into a homotopy cofibration sequence:

$$(6.10) \quad \operatorname{colim}_{K>H} \underline{\mathbf{X}}_k(G/K) \rightarrow \tilde{X}(H) \rightarrow \underline{\mathbf{X}}_H^H$$

(compare (6.6)). Note that the action of $N_G H$ on \mathcal{F}_k by conjugation defines a W_H -action on $\operatorname{colim}_{K>H} \underline{\mathbf{X}}_k(G/K)$.

In this case, for each $n \geq 1$, $e_n \in H^{n+2}(W_H; \pi_{n+1}F)$ is the n -th obstruction of Proposition 5.14 to interpolating a W_H -action on $\tilde{X}(H)$ in (6.10).

The two kinds of difference obstructions $(d_n) \in H^{n+1}(W_H; \pi_{n+1} \mathbf{BG}_{\tilde{\mathbf{X}}_H^H}^*)$ or $f_n \in H^{n+1}(W_H; \pi_{n+1}F)$ for distinguishing between different extensions of $\underline{\mathbf{X}}_k$ to $\underline{\mathbf{X}}_{k+1}$ are defined analogously.

We are now in a position to state our main result on realizing Bredon homotopy actions:

6.11. Theorem. *For any finite group G , a Bredon homotopy action $\langle \tilde{X}, (\alpha_H^*)_{H \leq G} \rangle$ can be realized by a G -space \mathbf{X} if and only if for each $k \geq 0$ and $\langle H \rangle \subseteq \hat{\mathcal{F}}_{k+1} \setminus \hat{\mathcal{F}}_k$, the two inductively defined $\langle H \rangle$ -sequences of obstructions $(c_n)_{n=1}^\infty$ and $(e_n)_{n=1}^\infty$ vanish. Moreover, two such realizations \mathbf{X} and \mathbf{X}' (by G -CW complexes) are G -homotopy equivalent if and only if the corresponding sequences of difference obstructions vanish.*

Proof. If $\langle \tilde{X}, (\alpha_H^*)_{H \leq G} \rangle$ can be realized by a G -space \mathbf{X} , the corresponding realizations $\underline{\mathbf{X}}_k$ were described in Example 6.8, and thus the obstructions must vanish.

To see that the vanishing of the $\langle H \rangle$ -obstructions suffices to extend $\underline{\mathbf{X}}_k$ (constructed by induction on the filtration (6.3)) to $\underline{\mathbf{X}}_{k+1}$, we start with $\underline{\mathbf{X}}_0(G/G) := \tilde{X}(G)$ (which we denote by \mathbf{Y}). To construct $\underline{\mathbf{X}}_1$, we must consider all maximal proper subgroups $M \in \hat{\mathcal{F}}_1$, which are of two types:

- (a) If $N_G M = M$, then $W_M = \{e\}$ and the correspondence $aM \mapsto M^a$ is a bijection between G/M and $\langle M \rangle$. In this case we change $\mathbf{Y} = \tilde{X}(G/G) \rightarrow \tilde{X}(G/M)$ into a cofibration $i : \mathbf{Y} \hookrightarrow \mathbf{Z}_{(M)}$ (with no group action), and form a diagram consisting of a copy $i_{(M^a)} : \mathbf{Y} \hookrightarrow \mathbf{Z}_{(M^a)}$ of i for each coset $M^a \in \langle M \rangle$, with $\underline{\mathbf{X}}_1((\tilde{\phi}_a^M)^{\text{op}})$ the homeomorphism identifying \mathbf{Z} with $\mathbf{Z}_{(M^a)}$ (relative to the fixed subspace \mathbf{Y}).
- (b) Otherwise $N_G M = G$, so $W_M = G/M$ and $\langle M \rangle$ is a singleton. We then apply Propositions 3.11 and 5.6 to obtain a W_M -action on $\mathbf{Z}_{(M)} \simeq \tilde{X}(G/M)$, extending the trivial action on $\mathbf{Y} := \tilde{X}(G/G)$. This is possible since we assume that the obstructions to doing so vanish. The action map $\zeta_{\mathbf{Z}_{(M)}} : G/M \rightarrow \mathcal{A}_{\mathbf{Z}_{(M)}}$ lifts to a G -action via the homomorphism $G \twoheadrightarrow G/M$.

Since all the conjugation G -actions we have described agree on \mathbf{Y} (where they are trivial), we obtain a diagram $\underline{\mathbf{X}}_1 : \mathcal{F}_1 \rightarrow \mathcal{T}$, whose restriction to Λ_1^{op} consists of the inclusions $\mathbf{Y} \hookrightarrow \mathbf{Z}_{(M)}$ for all $M \in \hat{\mathcal{F}}_1 \setminus \hat{\mathcal{F}}_{i-1}$.

At the k -th stage of the induction, we assume given a diagram $\underline{\mathbf{X}}_{k-1} : \mathcal{F}_{k-1} \rightarrow \mathcal{T}$ realizing \tilde{X} up to filtration $k-1$: in particular, for each $H \in \hat{\mathcal{F}}_k \setminus \hat{\mathcal{F}}_{k-1}$ we have a space $\underline{\mathbf{X}}_H := (\underline{\mathbf{X}}_{k-1})_H$ as in §6.4, on which $N_G H$ acts (by conjugation),

with $H \subseteq N_G H$ acting trivially. Thus \underline{X}_H has a W_H -action compatible with the structure maps of \underline{X}_{k-1} .

For each conjugacy class $\langle H \rangle \subseteq \widehat{\mathcal{F}}_k \setminus \widehat{\mathcal{F}}_{k-1}$, we choose an (arbitrary) fixed representative H . We then use Proposition 3.11 to lift the given pointed homotopy action of W_H on $\widetilde{\mathbf{X}}_H^H$ to a (free) pointed action on $\mathbf{X}_H^H \simeq \widetilde{\mathbf{X}}_H^H$ (which is possible, since the obstructions vanish). Next, use Proposition 5.6 to produce a W_H -interpolation of the given W_H -actions on \underline{X}_H and \mathbf{X}_H^H for the homotopy cofibration sequence (6.6). Denote the new W_H -space we have produced by $\mathbf{Z}_{(H)} \simeq \widetilde{X}(G/H)$. By Proposition 5.10, we may assume that the inclusion $i_{(H)} : \underline{X}_H \hookrightarrow \mathbf{Z}_{(H)}$ is W_H -equivariant (with respect to the given conjugation action on \underline{X}_{k-1}).

For any conjugate $H^a \in \langle H \rangle$, choose a fixed element $a \in G$ representing the coset $aN_G H \in G/N_G H \cong \langle H \rangle$, and let $\underline{X}_k(G/H^a) := \mathbf{Z}_{(H)}$. The W_{H^a} -action on $\underline{X}_k(G/H^a)$ is the composite of the action map $W_H \rightarrow \mathcal{A}_{\mathbf{Z}_{(H)}}$ with the isomorphism $(\rho_a^H)^{-1} : W_{H^a} \rightarrow W_H$ induced by $\rho_a^H : N_G H \rightarrow N_G H^a$ (conjugation by a).

We define $i_{H^a} : \underline{X}_{H^a} \hookrightarrow \mathbf{Z}_{(H)}$ to be the composite $i_{(H)} \circ (\widetilde{\phi}_a^H)^{\text{op}}$. This is W_{H^a} -equivariant because $(\widetilde{\phi}_a^H)^{\text{op}}$ is induced by ρ_a^H , so we have extended \underline{X}_{k-1} to a diagram $\underline{X}_k : \mathcal{F}_k \rightarrow \mathcal{T}$.

Note that if the difference obstructions $(d_n)_{n=1}^\infty$ for realizing the pointed homotopy W_H -action on $\widetilde{\mathbf{X}}_H^H$ vanish, we obtain lifts of $\mathbf{B}\alpha_H^* : \mathbf{B}W_H \rightarrow \mathbf{B}\mathcal{E}_{\widetilde{\mathbf{X}}_H^H}^*$ to homotopic maps $\Phi^* \sim (\Phi')^* : \mathbf{B}W_H \rightarrow \mathbf{B}\mathcal{G}_{\widetilde{\mathbf{X}}_H^H}^*$, and thus pointed Borel W_H -equivalences $\mathbf{X}_H^H \rightarrow (\mathbf{X}')_H^H$ (assuming both are W_H -CW complexes).

If also the difference obstructions $(f_n)_{n=1}^\infty$ for interpolating the W_H -actions vanish, we again have homotopic lifts $\Psi \sim \Psi' : \mathbf{B}W_H \rightarrow \mathbf{B}\mathcal{Q}_{f,g}$, so again the resulting W_H -spaces \mathbf{Z}_H and \mathbf{Z}'_H are Borel equivalent. But this implies that we have a weak equivalence of the resulting \mathcal{F}_{k+1} -diagrams \underline{X}_{k+1} and \underline{X}'_{k+1} , since all the structure maps which are not inclusions can be described in terms of the conjugation action of G . \square

6.12. Generalizations. The procedure described above extends to some infinite groups G , as long as we have a class function $\ell : \Lambda^G \rightarrow \kappa$ into some ordinal κ with $\ell(K) \preceq \ell(H)$ for $H \leq K$. In this case we have a filtration corresponding to (6.3) of length κ , and thus a transfinite inductive procedure as in the proof of Theorem 6.11.

For example, if $G = \mathbb{Z}$ then $\ell : \Lambda^G \rightarrow \omega + 1$ assigns to $n\mathbb{Z} \leq \mathbb{Z}$ the number of (not necessarily distinct) prime factors of n , with $\ell(\{0\}) = \omega$. On the other hand, there is no such function ℓ for $G = \mathbb{Z}^2$ or S^1 .

7. SOME SIMPLE CASES

The approach to realizing homotopy actions described here is quite complicated, in general, even for cyclic groups. Nevertheless, in certain cases the theory simplifies to some extent:

7.1. Free actions. A free action of G on \mathbf{X} is completely characterized by the fact that $\mathbf{X}^H = \emptyset$ for all $\{e\} \neq H \leq G$. In terms of a Bredon homotopy action $\langle \widetilde{X}, (\alpha_H^*)_{H \leq G} \rangle$, this means that $\widetilde{X}(H)$ is empty for $\{e\} \neq H$, so the obstructions of Theorem 6.11 all vanish vacuously except when $H = \{e\}$, where they reduce to the obstructions of [C].

7.2. Pointed free actions. A *pointed* free action on \mathbf{X} is characterized by having $\mathbf{X}^H = \{*\}$ for all $\{e\} \neq H \leq G$; in terms of a Bredon homotopy action this implies only that $\tilde{X}(H)$ is contractible for $\{e\} \neq H$, and that the maps $j_H : \tilde{\mathbf{X}}_H \rightarrow \tilde{X}(H)$ of (6.6) are all homotopy equivalences, so $\tilde{\mathbf{X}}_H^H$ is contractible, for all such H . Therefore, $\mathbf{BG}_{\tilde{\mathbf{X}}_H^H}^*$ is contractible, so once more the obstructions of Theorem 6.11 vanish except when $H = \{e\}$ (so $W_H = G$), where they reduce to the obstructions of Proposition 3.19

7.3. Semi-free actions. In a semi-free action all fixed points are global. In terms of a Bredon homotopy action this implies that the maps $j_H : \tilde{\mathbf{X}}_H \rightarrow \tilde{X}(H)$ are again homotopy equivalences for $\{e\} \neq H$, and thus $\tilde{\mathbf{X}}_H^H$ is contractible – but $\tilde{X}(H) \simeq \tilde{X}(G) =: \mathbf{Y}$ need not be contractible. However, we do have a trivial G -action on \mathbf{Y} . Thus we are left with two kinds of obstructions to realizing the given homotopy action:

- (a) Those of Proposition 3.19 for realizing the given pointed homotopy action of $W_H = G$ on $\mathbf{Z} := \tilde{\mathbf{X}}_H^H$ for $H = \{e\}$.
- (b) Those of Proposition 5.14 for interpolating the given G -actions in the homotopy cofibration sequence $\mathbf{Y} \rightarrow \tilde{X}(\{e\}) \rightarrow \mathbf{Z}$.

If both sets of obstructions vanish, we obtain the required semi-free G -action on a space $\mathbf{X} \simeq \tilde{X}(\{e\})$.

7.4. Diagrams of Eilenberg-Mac Lane spaces. A necessary condition in order for our obstruction theory to be effectively computable is that the (homotopy groups of) the spaces of self-equivalences of $\tilde{\mathbf{X}}_H$ and $\tilde{\mathbf{X}}_H^H$ in (6.6) are known for each $H \leq G$.

One simple case where this holds is when each of the above spaces is an Eilenberg-Mac Lane space $K := K(\pi, n)$, since in this case

$$(7.5) \quad \mathcal{G}_K \simeq K(\pi, n) \times \text{Aut}(\pi)$$

(as a monoid) is a semi-direct product of the Eilenberg-Mac Lane space itself and the discrete group $\text{Aut}(\pi)$, while $\mathcal{G}_K^* \simeq \text{Aut}(\pi) \cong \mathcal{E}_K^*$ is homotopically discrete (see [M1, Proposition 25.2]).

In this case there is no obstruction to lifting a pointed homotopy action of W_H on $\tilde{\mathbf{X}}_H^H$ to a strict action, by Corollary 3.17. If also $\pi_* \mathcal{G}_{\tilde{X}(H)}$ are known – e.g., if $\tilde{X}(H)$ is also an Eilenberg-Mac Lane space – then the homotopy groups of

$$(7.6) \quad \text{Map}(\tilde{\mathbf{X}}_H, \tilde{X}(H)) \quad \text{and} \quad \text{Map}(\tilde{X}(H), \tilde{\mathbf{X}}_H^H)$$

are known (by [T]), so we may determine the homotopy groups of \mathcal{P}_{j_H} and \mathcal{P}_{q_H} up to an extension from the pullback diagram (4.5), (7.5), and (7.6), respectively, and these determine the homotopy groups of \mathcal{Q}_{j_H, q_H} – and thus of the fibers F – up to extensions from and (5.5) and (5.13).

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